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STABILITY DESIGN AND CODE RULES FOR STRAIGHT TIMBER BEAMS

by

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Summary

The design rules of the Eurocode for lateral buckling are not general enough and not consistent and need to be revised. A general approach is therefore given of the buckling and twist-bend buckling problem of symmetrical profiles loaded in bending in the two main directions and at the same time in torsion and compression.
The model, according to the second order stress theory, gives an extension of the existing models by accounting for eccentrical lateral loading, for instance by purlin hangers, in combination with bending in the horizontal direction (wind loading etc.), with the influence of the initial eccentricities, the warping rigidity and the failure criterion.
The failure criterion is related to a linear behaviour until fracture, because the experimental strength of the beam is based on this approach.
Local buckling of thin webs and flanges is assumed to be prevented by stiffeners. For the stability calculation of this case, the Eurocode or [1] can be followed.
The derivation is based on an extension of the general differential equations of Chen and Atsuta [2], to eccentrically applied lateral loading. These equations can be modified to the form of those of Brüninghoff [6], with an equivalent torsional rigidity to account for the influence of warping and the Wagner effect. (The Wagner effect is the torsional moment appearing by the components of the normal stresses in a warped cross section).
The solution of the differential equations is done by the Galerkin method and results in a generalisation of the solution of [6].
Application of of the failure criterion gives comparable expressions as given by Larsen [4], [5], extended for eccentrically applied lateral loading.
The equations of Brüninghoff and Larsen are thus special cases of the general expression, and are verified by tests for these and other cases [3], [5], [6] (and thesis work Stevin-laboratory).

Conclusion

The theory gives an extension of the existing methods to the general loading case of eccentrically loading in all directions (double bending with torsion and compression) and predicts low instability values for a short beam with a low warping stiffness, loaded laterally on the compression side. The existing design methods are unsafe for this case and it can be shown that the warping stiffness of rectangular beams is not neglectable in this situation.
The method provides more general and better rules for the Eurocode and will be proposed to replace art. 5.1.6, 5.1.10, 5.2.6 (see appendix 2).

Notation

M_x, M_y	bending moment about the x-axis and y-axis
M'_x	first derivative of M_x to z, along the axis of the beam
M_t	torsional moment about the beam-axis
M_u	ultimate moment for failure = $f_b \cdot W$ (ultimate stress times moment of rigidity)
M_k	theoretical twist-bend moment for pure bending and compression = $\sqrt{F_{ey} \cdot GI_m}$
M_c	theoretical twist-bend moment for lateral loading = $M_k \cdot \alpha$
M_{c0}	is M_c for F = 0, so for bending without compression
M_{lat}	real twist-bend moment with initial eccentricities (lateral buckling)
F	normal force
F_e	Eulerian buckling load = $\pi^2 EI/L^2$
F_c	buckling load (with initial eccentricities)
F_u	Ultimate compressive force = $f_c \cdot A$ (ultimate normal stress times section-area)
F_t	twist buckling force = $GI_t \cdot \left(1 + (\pi^2 \cdot EI_w/(GI_t \cdot L^2))\right)/\left(I_x + I_y\right)$
u, v	deformations in resp. x- and y- axis
u', v'	differentiation of u and v with respect to z
φ	rotation about the z- axis
α	factor due to the eccentricity of the lateral load
λ	is: $L/i = L \cdot \sqrt{A/I}$ slenderness
σ_d	compressive stress; f_c is the compressive strength
σ_b	bending stress; f_b is the bending strength
EI_x. EI_y	bending rigidity about resp. the x- axis and y- axis
EI_w	warping rigidity
K	Wagner effect = $-F \cdot (I_x + I_y)/A$
GI_v	equivalent torsional rigidity for high beams = $GI_t \cdot (1 - \frac{F}{F_t}) \cdot \left(1 + \frac{\pi^2 \cdot EI_w}{GI_t \cdot L^2}\right)$
GI_t	Torsional rigidity (St. Venant)
GI_m	equivalent torsional rigidity = $GI_v \cdot (1 - \frac{F}{F_{ex}}) \cdot \frac{1}{1 - EI_y/EI_x}$
W_x, W_y	moment of rigidity
e, s	eccentricity of the lateral loading
p, q	lateral loading
r	is: W/A, radius of rigidity
L	span, or effective buckling length
A	Area of the cross-section of the beam

1. Introduction

The stability design of the Eurocode is not general and consistent enough. For instance, in the Eurocode the warping rigidity is neglected for free beams without a horizontal bracing. For braced beams however the torsional rigidity is neglected. Further the initial eccentricities are regarded for braced beams and neglected for free beams although the reversed would have been better. The given influence of the point of application of the lateral loading on l_{ef} applies only for long beams. So a more general approach is necessary. However the known calculation methods for twist-bend buckling are incomplete and often mutually contradictory and need to be extended.

By Chen and Atsuta [2], general equations are given for thin walled beams. However solutions are only giiven for pure bending with compression (thus without lateral loading).

The influence of lateral loading is given by Halasz and Cziesseilski [3], however without initial eccentricities and without normal loading. The influence of warping is also not regarded there and thus there is no distinction between I-beams and box-beams. This is well done in [4] for I-beams, while the warping rigidity of rectangular- and box-beams is neglected. By Larsen [5], general equations are given for the case of pure bending and compression, including the influence of initial eccentricities and the failure criterion. The warping rigidity is however neglected (as also is done by all authors for rectangular beams) although there is accounted for warping deformation by the reduction of the torsional rigidity by the negative Wagner effect. This means that it is assumed that there is an unrestrained warping. However restraint warping and warping rigidity is always assumed to exist for thin-webbed beams and trusses (beams with low torsional rigidity), for instance in most regulations, because the twist-bend buckling of these profiles is calculated from the column buckling of the compressed flange, what is equivalent to a dominating warping rigidity.

By Brüninghoff [6], the influence of the eccentricity of the lateral loading and the initial eccentricities are regarded for high rectangular beams. However the failure criterion is not regarded and also the warping rigidity neglected, as is only right for long rectangular beams. Because comparable general equations, including the influence of warping and the failure criterion, for the general loading case are lacking for beams and for thin-webbed beams, the derivation is given here.

2 Stability of a symmetrical beam loaded in compression and double bending

2.1 General differential equations

From equilibrium of a deformed element, the general differential equations are given by Chen and Atsuta ([2], eq.(2.179a)). For symmetrically beams these simplify to (see notations):

$$EI_x \cdot v'''' + F \cdot v'' + (\varphi \cdot M_y)'' - u''' \cdot M_t - 2 \cdot u'' \cdot M_t' + M_x'' = 0 \quad (1)$$

$$EI_y \cdot u'''' + F \cdot u'' + (\varphi \cdot M_x)'' - v''' \cdot M_t - 2 \cdot v'' \cdot M_t' + M_y'' = 0 \quad (2)$$

$$EI_w \cdot \varphi'''' - (GI_t + K) \cdot \varphi'' + u'' \cdot M_x + v'' \cdot M_y - u \cdot M_x'' - v \cdot M_y'' + M_t' = 0 \quad (3)$$

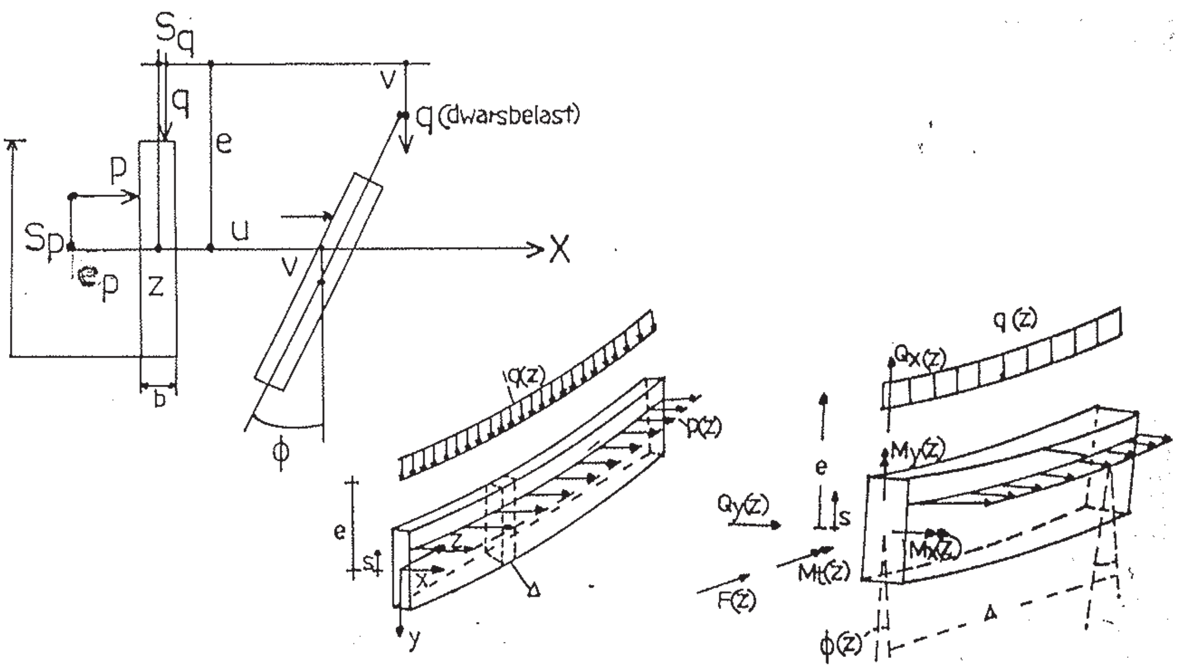


fig. 1

Further simplification is possible by omitting small terms. This can be seen by using the first term of the Fourier expansion of the variables.

For simply supported beams is for instance: $u = \bar{u} \cdot \sin(\pi \cdot z/L)$ and $M_t = \bar{M}_t \cdot \cos(\pi \cdot z/L)$ and the term: $u'' \cdot M_t'$ of eq.(1) has a maximum value of order: $\bar{u} \cdot \bar{M}_t \cdot \pi^3/L^3$. Also the maximum value of $u''' \cdot M_t$ is of this order. As shown later, eq.(5), the top-value of $2 \cdot u'' \cdot M_t'$ is: $(\pi^2/L^2) \cdot \bar{u} \cdot (\bar{p} \cdot h + \bar{q} \cdot b)$, what is neglectable with respect to: $-\bar{M}_x'' = \bar{q}$ in eq.(1).

In the same way, it can be shown that if $q = 0$, the term: $2 \cdot u'' \cdot M_t'$ is small with respect to the terms $(\varphi \cdot M_y)''$ and $EI_x \cdot v''''$ and the terms with M_t and M_t' can be ommitted in eq.(1) and for the same reason also in eq.(2).

The values $v'' \cdot M_y$ and $v \cdot M_y''$ are also comparable and equal to: $\frac{\pi^2}{L^2} \cdot \bar{v} \cdot \bar{M}_y \cdot \sin\left(\frac{2\pi}{L} \cdot z\right)$ and in the same way is: $u'' \cdot M_x = u \cdot M_x'' = -q \cdot u$ in eq.(3).

From fig. 1, it follows that the increase of the torsional moment per unit length is:

$$-M_t' = p \cdot s_p + q \cdot s_q - p \cdot v + q \cdot u + p \cdot \varphi \cdot e_p + q \cdot \varphi \cdot e_q \quad (4)$$

So $2 \cdot u'' \cdot M_t' \mathrel{\hat{=}} 2 \cdot u \cdot (\pi^2/L^2) \cdot (p \cdot s_p + q \cdot s_q - p \cdot v + q \cdot u + p \cdot \varphi \cdot e_p + q \cdot \varphi \cdot e_q)$, and for high eccentricities, for instance: $s_p = h/2$ and $s_q = b/2$, the terms: $p \cdot h/2 + q \cdot b/2$ dominate (because $v \ll h/2$; $u \ll b/2$; $\varphi \cdot e_p \ll h/2$; $\varphi \cdot e_q \ll b/2$) and $2 \cdot u'' \cdot M_t'$ becomes:

$$2 \cdot u'' \cdot M_t' \approx u \cdot (\pi^2/L^2) \cdot (p \cdot h + q \cdot b) \tag{5}$$

For small eccentricities, for instance $s_p = s_q = 0$, this term is much smaller and it is seen that this term is always neglectable.
In eq.(3), φ'''' can be replaced by: $\varphi'''' = -(\pi^2/L^2) \cdot \varphi''$, and eq.(3) can be written in the same form as given in [6]:

$$(EI_w \cdot (\pi^2/L^2) + GI_t + K) \cdot \varphi'' - u'' \cdot M_x - M_t' - q \cdot u = 0 \tag{6}$$

According to eq.(4) is:

$$-M_t' - q \cdot u = p \cdot s_p + q \cdot s_q - p \cdot v + p \cdot \varphi \cdot e_p + q \cdot \varphi \cdot e_q = p \cdot s_v + q \cdot \varphi \cdot e_v - p \cdot v, \text{ with:}$$

$$s_v = s_p \cdot \left(1 + \frac{q \cdot s_q}{p \cdot s_p}\right) \text{ and } e_v = e_q \cdot \left(1 + \frac{p \cdot e_p}{q \cdot e_q}\right)$$

With: $GI_v = \frac{\pi^2}{L^2} \cdot EI_w + GI_t + K = GI_t \cdot \left(1 + \frac{\pi^2}{L^2} \cdot \frac{EI_w}{GI_t} - \frac{F(I_x + I_y)}{GI_t \cdot A}\right) = GI_t \cdot \left(1 + \frac{\pi^2 EI_w}{L^2 \cdot GI_t}\right) \cdot \left(1 - \frac{F}{F_t}\right)$,

where: $F_t = \frac{GI_t \cdot A}{I_x + I_y} \cdot \left(1 + \frac{\pi^2 EI_w}{L^2 \cdot GI_t}\right)$ is the twist buckling force, eq.(6) can be written:

$$GI_v \cdot \varphi'' - u'' \cdot M_x + p \cdot s_v + q \cdot \varphi \cdot e_v - p \cdot v = 0 \tag{7}$$

For high beams the term: $p \cdot v$ is small and can be neglected is eq.(7).
For high beams, $I_x \gg I_y$ and thus $p \ll q$, is in eq.(1) also the term: $(\varphi \cdot M_y)''$ neglectable because:

$$(\varphi \cdot M_y)'' \approx 4 \cdot p \cdot \varphi \ll M_x'' = -q.$$

However in eq.(2) is, for high beams, $(\varphi \cdot M_x)'' \approx 4 \cdot q \cdot \varphi$ not always of lower order than: $-p$ or $EI_y \cdot u''''$ and can only be neglected for low beams. So for high beams, eq.(1) to eq.(3) are:

$$EI_x \cdot v'''' + F \cdot v'' - q = 0 \tag{1'}$$

$$EI_y \cdot u'''' + F \cdot u'' + (\varphi \cdot M_x)'' - p = 0 \tag{2'}$$

$$GI_v \cdot \varphi'' - u'' \cdot M_x + p \cdot s_v + q \cdot \varphi \cdot e_v = 0 \tag{3'}$$

For low beams, where I_x and I_y are not far apart, eq.(1) to eq.(3) become:

$$EI_x \cdot v'''' + F \cdot v'' - q = 0 \tag{1''}$$

$$EI_y \cdot u'''' + F \cdot u'' - p = 0 \tag{2''}$$

$$GI_v \cdot \varphi'' + q \cdot u - p \cdot v + p \cdot s_p + q \cdot s_q + p \cdot e_p \cdot \varphi + q \cdot e_q \cdot \varphi = 0 \tag{3''}$$

Now is: $q \cdot u - p \cdot v = q \cdot u \cdot \left(1 - \frac{p \cdot v}{q \cdot u}\right) = q \cdot u \cdot \frac{1 - EI_y/EI_x}{1 - F/F_{ex}}$, because according to eq.(1'') and (2''):

$$\frac{p\cdot v}{q\cdot u} = \frac{(EI_y\cdot\pi^4/L^4 - F\cdot\pi^2/L^2)\cdot u\cdot v}{(EI_x\cdot\pi^4/L^4 - F\cdot\pi^2/L^2)\cdot u\cdot v} = \frac{EI_y}{EI_x}\cdot\frac{1 - F/F_{ey}}{1 - F/F_{ex}} = \frac{EI_y/EI_x - F/F_{ex}}{1 - F/F_{ex}}$$

where: $F_{ex} = (\pi^2/L^2)\cdot EI_x$ and: $F_{ey} = (\pi^2/L^2)\cdot EI_y$ are the Eulerian buckling loads.
Eq.(3") can now be written with $q\cdot u = -M_x''\cdot u = -M_x\cdot u''$:

$$\frac{GI_v\cdot(1 - F/F_{ex})}{1 - EI_y/EI_x}\cdot\varphi'' - M_x\cdot u'' + \frac{1 - F/F_{ex}}{1 - EI_y/EI_x}\cdot(p\cdot s_v + q\cdot e_v\cdot\varphi) = 0 \tag{8}$$

or:

$$GI_m\cdot\varphi'' - M_x\cdot u'' + p\cdot s_m + q\cdot e_m\cdot\varphi = 0 \tag{8'}$$

where: GI_v, s_v and e_v are multiplied by: $\dfrac{1 - F/F_{ex}}{1 - EI_y/EI_x}$, to get resp. GI_m, s_m, e_m.

Equation (8') has the same form as eq.(3') and can safely also be applied for high beams, where $I_y \ll I_x$, because then also: $F \ll F_{ex}$ and eq.(8') approaches eq.(3').
Equations (8'), (1') and (2') can now be used as well as for high beams ($I_x \gg I_y$) as for low beams ($I_y \rightarrow I_x$) because this system turns to eq.(1'), (2'), (3') for high beams and to eq.(1"), (2"), (3") for low beams.
If now the initial eccentricities u_0, v_0 and φ_0 are introduced, then the general applicable differential equations for $I_x > I_y$ are:

$$EI_x\cdot(v'''' - v_0'''') + F\cdot v'' + M_x'' = 0 \tag{1'''}$$

$$EI_y\cdot(u'''' - u_0'''') + F\cdot u'' + (\varphi\cdot M_x)'' + M_y'' = 0 \tag{2'''}$$

$$GI_m\cdot(\varphi'' - \varphi_0'') - M_x\cdot u'' + p\cdot s_m + q\cdot e_m\cdot\varphi = 0 \tag{3'''}$$

where eq.(3''') is at the safe side if φ_0 is important.
Equations (2''') and (3''') have the same form as those of [6]. The differences are the equivalent eccentricities and rigidity GI_m (in stead of GI_t) due to warping effects.

2.2 Solution of the differential equations

Equation (1''') is directly solvable. For instance with: $v = \bar{v}\cdot\sin(\pi\cdot x/L)$; $v_0 = \bar{v}_0\cdot\sin(\pi x/L)$ and $q = \bar{q}\cdot\sin(\pi x/L)$ is eq.(1'''):

$$\bar{v} = \frac{F_{ex}\bar{v}_0 + \bar{M}_x}{F_{ex} - F} \qquad \text{with: } \bar{M}_x = \bar{q}\cdot L^2/\pi^2 \text{ and } F_{ex} = \pi^2\cdot EI_x/L^2.$$

Because $M_{x,F} = -EI_x\cdot(v - v_0)''$, is: $\bar{M}_{x,F} = \dfrac{\pi^2}{L^2}\cdot EI_x\cdot(\bar{v} - \bar{v}_0)$ and:

$$M_{x,F} = \frac{M_x + F\cdot v_0}{1 - F/F_{ex}} \tag{9}$$

The solution of eq.(2''') and eq.(3''') can be found by the Galerkin method:

For a given differential equation: $L(u) = 0$, in wich L is a differential operator, a solution is assumed in a series form as: $u(z) = \sum_1^n a_i \cdot f_i(z)$, where $f_i(z)$ are known functions wich satisfy all boundary conditions (both geometric and static), then the coefficients a_i's can be obtained from the n conditions:

$$\int_0^L \bar{L}(u)\cdot f_j(z)\cdot d(z) = \int_0^L \bar{L}\Big(\sum_i a_i\cdot f_i(z)\Big)\cdot f_j(z)\cdot d(z) = 0 \qquad (10)$$

This gives n algebraic simultaneous equations for determination of a_i to a_n.
The application of this method for eq.(2''') and eq.(3'''), is given in the appendix. For the f_i's, the first expanded term of the Fourier sine series are taken:
$u = \bar{u}\cdot\sin(\pi z/L)$; $\varphi = \bar{\varphi}\cdot\sin(\pi z/L)$; $u_0 = \bar{u}_0\cdot\sin(\pi z/L)$; $\varphi_0 = \bar{\varphi}_0\cdot\sin(\pi z/L)$;
$p = \bar{p}\cdot\sin(\pi z/L)$; $q = \bar{q}\cdot\sin(\pi z/L)$; $M = \bar{M}\cdot\sin(\pi z/L)$.
Multiplication by $f_j = \sin(\pi z/L)$ in eq.(10) and integration over the length L of the beam, gives two solvable equations in $\bar{u}$ and $\bar{\varphi}$ (see appendix).
Substitution of: $u = \bar{u}\cdot\sin(\pi z/L)$ in: $M_y = -EI_y\cdot(u - u_0)''$, or: $\bar{M}_y = \frac{\pi^2}{L^2}\cdot EI_y\cdot(\bar{u} - \bar{u}_0)$ gives:

$$\bar{M}_{y,F} = \frac{F\cdot\bar{u}_0\cdot\left(1 - \frac{q\cdot L^2}{\pi^2}\cdot\frac{e_m}{GI_m}\right) + \bar{u}_0\cdot F\cdot e_y\cdot\frac{M_x^2}{M_k^2} + \bar{\varphi}_0\cdot M_x + \bar{M}_y\cdot\left(1 + \frac{M_x\cdot s_m\cdot p\cdot L^2}{\pi^2\cdot GI_m\cdot\bar{M}_y} - \frac{e_m\cdot q\cdot L^2}{\pi^2\cdot GI_m}\right)}{\left(1 - \frac{q\cdot L^2\cdot e_m}{\pi^2\cdot GI_m}\right)\cdot\left(1 - \frac{F}{F_{ey}}\right) - \frac{M_x^2}{M_k^2}} \qquad (11)$$

in wich: $q = \frac{8}{3\cdot\pi}\cdot\bar{q}$; $M_x = \frac{8}{3\cdot\pi}\cdot\bar{M}_x$ and $M_k = \sqrt{F_{ey}\cdot GI_m}$, with: $F_{ey} = \frac{\pi^2}{L^2}\cdot EI_y$.

If M_x and M_y are only due to lateral loading resp. q and p, then: $M_x = \frac{q\cdot L^2}{\pi^2}$ and $\bar{M}_y = \frac{\bar{p}\cdot L^2}{\pi^2}$ and eq.(11) becomes (with omission of the top-bar-sign):

$$M_{y,F} = \frac{F\cdot u_0\cdot\left(1 - \frac{e_m\cdot M_x}{GI_m}\right) + F_{ey}\cdot u_0\cdot\frac{M_x^2}{M_k^2} + \varphi_0\cdot M_x + M_y\cdot\left(1 + \frac{M_x}{GI_m}\cdot(s_m - e_m)\right)}{\left(1 - \frac{F}{F_{ey}}\right)\cdot\left(1 - \frac{e_m\cdot M_x}{GI_m}\right) - \left(\frac{M_x}{M_k}\right)^2} \qquad (11')$$

If the beam is only loaded by boundary moments, for instance: $M_x = M_0$ at $z = 0$, and for $z = L$: $M_x = M_L$, then $M_x'' = 0$ and $q = 0$ has to be taken in eq.(11).
For a combination of an eccentrical lateral loading: q with boundary moments: $M_{x,m}$ is, for a proportional loading increase, the ratio: $M_{x,m}/q$ constant and eq.(11') is generally applicable if e_m is corrected to e'_m according to:

$$e'_m = e_m\cdot\frac{q\cdot L^2/\pi^2}{M_x} = e_m\cdot\frac{q\cdot L^2/\pi^2}{M_{x,m} + q\cdot L^2/\pi^2} = \frac{e_m}{1 + M_{x,m}\cdot\pi^2/(q\cdot L^2)} = \frac{e_m}{1 + \bar{M}_{x,m}\cdot\pi^2/(\bar{q}\cdot L^2)}$$

and s_m according to:

$$s'_m = \frac{s_m}{1 + M_{y,m}\cdot\pi^2/(p\cdot L^2)}$$

$\bar{M}_{x,m}$ is the top-value of the first expanded term of the Fourier expansion of the moment surface due to the boundary moments.
The same applies for the moments about the y-axis: $M_{y,m}$.
The equations (9) and (11') are thus the wanted solutions of the system eq.(1'''), (2''') and (3''').

2.3 Simplification and asymptotecal values of twist-bend buckling.

In eq.(11') is: $M_k = \sqrt{F_{ey} \cdot GI_m}$, the theoretical twist-bend moment for pure bending.
This means that for: $e = s = u_0 = \varphi_0 = F = 0$, is:

$$M_{y,F} = \frac{M_y}{1 - \left(\frac{M_x}{M_k}\right)^2}$$ and this becomes very large if M_x approaches M_k.

According to measurements of Larsen [5], $\varphi_0 \cdot M_x$ in eq.(11') is neglectable and then in eq.(11'), the nominator and denominator can be devided by: $1 - (M_x \cdot e_m)/GI_m$ and eq.(11') becomes:

$$M_{y,F} = \frac{u_0 \cdot \left(F + F_{ey} \cdot \frac{M_x^2}{(M'_c)^2}\right) + M_y \cdot \left(1 + \frac{M_x \cdot F_{ey} \cdot s_m}{(M'_c)^2}\right)}{1 - \frac{F}{F_{ey}} - \frac{M_x^2}{(M'_c)^2}} \qquad (11'')$$

with: $(M'_c)^2 = M_k^2 \cdot \left(1 - \frac{e_m \cdot M_x}{GI_m}\right)$. This term $(M'_c)^2$ can be written as:

$$(M'_c)^2 = M_c^2 \cdot \frac{1 - \frac{e_m \cdot M_x}{GI_m}}{1 - \frac{e_m \cdot M_c}{GI_m}} \approx M_c^2 \cdot \left(1 - \frac{e_m \cdot M_x}{GI_m} + \frac{e_m \cdot M_c}{GI_m}\right) \geq M_c^2$$

where M_c follows from the zero value of the denominator of eq.(11'). So M_c is the reduced theoretical twist-bend buckling moment by lateral loading and normal force because for $M_x \rightarrow M_c \cdot \sqrt{1 - F/F_{ey}}$, $M_{y.F}$ becomes very large.
For high values of M_x, M_x approaches M_c, so $M'_c \rightarrow M_c$ and for high values of F is M_x small and so $(M_x/M'_c)^2$ is very small and the deviation between M'_c and M_c has a little influence. So with a small neglection on the safe side, M'_c can be replaced by M_c and is eq.(11''):

$$M_{y,F} = \frac{u_0 \cdot \left(F + F_{ey} \cdot \frac{M_x^2}{(M_c)^2}\right) + M_y \cdot \left(1 + \frac{M_x \cdot F_{ey} \cdot s_m}{(M_c)^2}\right)}{1 - \frac{F}{F_{ey}} - \frac{M_x^2}{(M_c)^2}} \qquad (11''')$$

As defined above, M_c follows from the equation:

$$M_c^2 + \frac{e_m \cdot M_k^2}{GI_m} \cdot M_c - M_k^2 = 0 \tag{12}$$

The resolution of eq.(12) is:

$$M_c = \sqrt{F_{ey} \cdot GI_m} \cdot \left(\sqrt{\left(\frac{e_m}{2}\right)^2 \cdot \frac{F_{ey}}{GI_m} + 1} - \sqrt{\left(\frac{e_m}{2}\right)^2 \cdot \frac{F_{ey}}{GI_m}} \right) = M_k \cdot \alpha \tag{13}$$

So the influence of the lateral loading can be accounted for by a constant α.

The real theoretical twist-bend buckling moment that makes the denominator of (11') zero $M_{c,F} = M_c \cdot \sqrt{1 - F/F_{ey}}$ is:

$$M_{c,F} = \frac{\pi}{L} \cdot \sqrt{\frac{EI_y \cdot GI_t \cdot \left(1 + \frac{\pi^2}{L^2} \cdot \frac{EI_w}{GI_t}\right) \cdot \left(1 - \frac{F}{F_t}\right) \cdot \left(1 - \frac{F}{F_{ey}}\right) \cdot \left(1 - \frac{F}{F_{ex}}\right)}{1 - \frac{EI_y}{EI_x}}} \cdot$$

$$\cdot \left[\sqrt{\left(\frac{e_{m0}}{2}\right)^2 \cdot \frac{F_{ey}}{GI_t} \cdot \frac{(1 - F/F_{ex}) \cdot (1 - F/F_{ey})}{1 - F/F_t} + 1} \quad + \right.$$

$$\left. - \sqrt{\left(\frac{e_{m0}}{2}\right)^2 \cdot \frac{F_{ey}}{GI_t} \cdot \frac{(1 - F/F_{ex}) \cdot (1 - F/F_{ey})}{1 - F/F_t}} \right] \tag{14}$$

For pure bending, $e_{m0} = 0$, this is a well known equation from theory (see for instance [2]). The index 0 of e_{m0} means that F in the expression for e_m is zero.

2.4 Stress criterion

The maximal stresses in the beam due to the moments $M_{x,F}$ and $M_{y,F}$, wich contain the second order effects, have to satisfy the stress criterium for failure. A reasonable approximation of this criterion, (see [4]) is:

$$\frac{\sigma_c}{f_c} + \frac{\sigma_b}{f_b} \leq 1 \tag{15}$$

with: $\sigma_b = (M_x/W_x) + (M_y/W_y)$, and $\sigma_c = F/A$ and f_c and f_b are resp. the compression and bending strengths.

This criterion is especially right for wet, high grade wood of smaller dimensions and is safe for dry lower grade timber (for combinations of bending and compression).

Substitution of eq.(9) and eq.(11''') in eq.(15) gives:

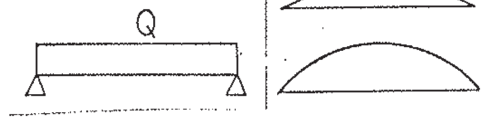
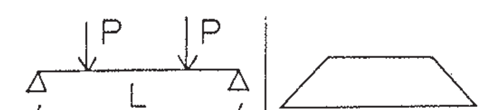
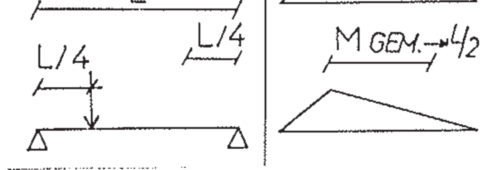
$$\frac{F}{f_c \cdot A} + \frac{M_x + F \cdot v_0}{f_b \cdot W_x \cdot (1 - \frac{F}{F_{ex}})} + \frac{u_0 \cdot \left(F + F_{ey} \cdot \frac{M_x^2}{M_c^2}\right) + M_y \cdot \left(1 + \frac{M_x \cdot s_m \cdot F_{ey}}{M_c^2}\right)}{f_b \cdot W_y \cdot \left(1 - \frac{F}{F_{ey}} - \frac{M_x^2}{M_c^2}\right)} \leq 1 \qquad (15')$$

This equation has the form of the equation of Larsen [5]. However it contains now the influence of warping, normal force and eccentrical lateral loading.

M_x in this equation stands for $\frac{8}{3 \cdot \pi} \cdot \bar{M}_x$, where $\bar{M}_x$ is the top-value of the first term of the Fourier series expansion of the moment area. M_y is equal to $\bar{M}_y$.

In stead of a Fourier expansion the following approximation is possible. The value: $\frac{8}{3 \cdot \pi} \cdot \bar{M}_x$

Table 1

loading	bending moment	$\frac{8}{3 \cdot \pi} \cdot \bar{M} =$ $= M_{max}/\rho$	exact value M/ρ	
		$\frac{M}{0.93}$	$\frac{M}{1}$	$\frac{M}{1}$
		$\frac{M}{1.85}$	$\frac{M}{1.75}$	$\frac{M}{1.67}$
		$\frac{PL/4}{1.45}$	$\frac{PL/4}{1.35}$	$\frac{PL/4}{1.33}$
		$\frac{q \cdot L^2/8}{1.14}$	$\frac{q \cdot L^2/8}{1.13}$	$\frac{qL^2/8}{1.09}$
		$\frac{PL/4}{1.03}$	$\frac{PL/4}{1.04}$	$\frac{PL/4}{1}$
		$\frac{3PL/16}{1.54}$	$\frac{3PL/16}{1.44}$	$\frac{3PL/16}{1.33}$
		1st expanded insufficient	$M \cdot (0.6 + 0.4 \cdot \kappa)$ $\geq 0.4 \cdot M$	

can be replaced by the mean moment of the middle half (at the largest buckling deformation) of the beam (with a minimum of $0.4 \cdot M_{max}$ if the moment changes sign along the beamlength and the beam buckles with double curvature).

So: $$\frac{M}{\rho} \approx \int_{-L/4}^{+L/4} \frac{M \cdot dx}{L/2} \quad (= M_{m,L/2})$$ given in the last column of the table.

With this, a simple rule is given for the values of ρ.
Eq.(15') can be written in the well known form:

$$\frac{F}{F_u} + \frac{n_x}{n_x - 1} \cdot \frac{F \cdot v_0 + M_x}{M_{ux}} + \frac{n'_y}{n'_y - 1} \cdot \frac{F' \cdot u_0 + M'_y}{M_{uy}} \leq 1 \qquad (15'')$$

with: $n_x = \frac{F_{ex}}{F}$; $n'_y = \frac{F_{ey}}{F'}$ with $F' = F + \frac{M_x^2}{M_c^2} \cdot F_{ey}$ and: $M'_y = M_y \cdot \left(1 + \frac{M_x \cdot s_m \cdot F_{ey}}{M_c^2}\right)$

3 Simplified design equations

Equation (15') is general applicable and simply programmable. For practice, possibly still more simply equations are desired.
Simplification is possible by expressing the combined loading cases in the instability equation in expressions for in plane buckling by compression and lateral buckling by bending alone. Then the instability equation turns into the so called interaction equation. This interaction equation and also the equations for in plane buckling and lateral buckling can then be simplified. For comparison, the usual cases of the codes will first be regarded.

3.1 In plane buckling

Useally the case of a lateral supported beam is in the regulations. For this case is:
$u_0 = \varphi_0 = 0$; $M_y = 0$; $F_{ey} \rightarrow \infty$ and $M_c \rightarrow \infty$.
Eq.(15') becomes:

$$\frac{F}{F_u} + \frac{M_x + F \cdot v_0}{M_{ux} \cdot (1 - F/F_{ex})} = 1 \qquad (16)$$

For $M_x = 0$, is: $F = F_c$ (centrically loaded beam) and the equation is:

$$\frac{F_c}{F_u} + \frac{F_c \cdot v_0}{M_u \cdot (1 - F_c/F_{ex})} = 1 \qquad (17)$$

F_c can be resolved of this equation and is:

$$\frac{F_c}{F_u} = \frac{1}{2}\cdot\left\{\left(1 + \frac{F_{ex}}{F_u} + \frac{F_{ex}\cdot v_0}{M_{ux}}\right) - \sqrt{\left(1 + \frac{F_{ex}}{F_u} + \frac{F_{ex}\cdot v_0}{M_{ux}}\right)^2 - \frac{4\cdot F_{ex}}{F_u}}\right\} \qquad (18)$$

The same equation applies for an unsupported high beam or column, buckling in the weak direction. Then the index: x has to be replaced by: y.

Eq.(18) is the same as eq.(5.1.10 g) of the Eurocode.

Introducing: $k_E = F_{ex}/F_u$, and: $v_0/r = \eta\cdot L/i = \eta\cdot\lambda$, or $F_{ex}\cdot v_0/M_{ux} = k_E\cdot\eta\cdot\lambda\cdot f_c/f_b$ and $k_{col} = F_c/F_u$ then eq.(18) is:

$$k_{col} = 0.5\cdot\left\{\left(1 + \left(1 + \eta\cdot\lambda\cdot\frac{f_c}{f_b}\right)\cdot k_E\right) - \sqrt{\left(1 + \left(1 + \eta\cdot\lambda\cdot\frac{f_c}{f_b}\right)\cdot k_E\right)^2 - 4\cdot k_E}\right\} \qquad (18')$$

This was an earlier proposal for the Eurocode.

From eq.(17), it can be seen that for a short test-specimen, when $F_{ex} \rightarrow \infty$, the strength is:

$$F_{c0}\cdot(1 + v_0\cdot F_u/M_{ux}) = F_u = F_{c0}(1 + \eta\cdot\lambda_0\cdot f_c/f_b) = F_{c0}\cdot(1 + c\cdot\eta) \qquad (17')$$

If there is accounted for the slenderness and initial eccentricity of the test-specimen for compression, by: $f_c = f_{c0}\cdot(1 + 20\cdot\eta)$, then, with: $k'_{col} = \sigma_c/f_{c0}$, and $k_{eu} = k_E\cdot(1 + 20\cdot\eta) = F_{ex}/F_{c0}$, eq.(18') becomes, according to the new Eurocode:

$$k'_{col} = 0.5\cdot\left\{ 1 + \left(1 + \eta\cdot\lambda\cdot\frac{f_{c0}}{f_m}\cdot(1 + 20\cdot\eta)\right)\cdot\frac{k_{eu}}{1 + 20\cdot\eta} + \right.$$

$$\left. - \sqrt{\left(1 + \left(1 + \eta\cdot\lambda\cdot\frac{f_c}{f_b}\cdot(1 + 20\cdot\eta)\right)\cdot\frac{k_{eu}}{1 + 20\cdot\eta}\right)^2 - \frac{4\cdot k_{eu}}{1 + 20\cdot\eta}}\right\}\cdot(1 + 20\cdot\eta) \qquad (18'')$$

Because $\lambda \approx 20$ for the test-specimen in compression the term $20\cdot\eta$ in the expressions above can better be replaced by $(20\cdot\eta\cdot f_c/f_b)/(1 - 1/k_E)$.

In the Dutch code T.G.B. 1972 is $v_0 = (0,1 + \lambda/200)\cdot r$ and eq.(17') is with $\lambda = 0$:

$$\frac{F_{c0}}{F_u} = 1 - \frac{v_0\cdot F_{c0}}{M_{ux}} = 1 - 0,1\cdot r\cdot\frac{f_c}{f_b}\cdot\frac{A}{W} = 1 - 0,1\cdot 0,75 = 1 - 0,075 = 0,925$$

where $f_c/f_b = 0,75$. So: $F_u = F_{c0}/0,925$ is used and $k_E = \pi^2\cdot E/(f_c\cdot\lambda\cdot 3,6)$. This gives comparable results as in the Eurocode. The condition of limited deformation is not used in the Eurocode because the strength condition is the only measure for safety.

For buckling and lateral buckling further simplifications are possible by re-arranging the terms of the equations for short beams as well as for slender beams in the form:

$$a = 1 - \frac{b}{1 - c} \quad \text{with } b \ll 1 \text{ and } c \ll 1$$

making the conservative approxiation possible:

$$a = \left(1 - \frac{b}{1-c}\right)\cdot \frac{1 + \frac{b}{1-c}}{1 + \frac{b}{1-c}} = \frac{1 - \left(\frac{b}{1-c}\right)^2}{1 + \frac{b}{1-c}} \approx \frac{1}{1 + \frac{b}{1-c}}$$

This too conservative value of "a" can partly be corrected by:

$$a \approx \frac{1}{1 + \frac{b}{1-c}\cdot\frac{1+c}{1+c}} = \frac{1}{1 + \frac{b + bc}{1 - c^2}} \approx \frac{1}{1 + b + bc_{max}}$$

If this is done for eq.(17) than eq.(18') may be replaced by:

$$k_{col} = \frac{1}{1 + (f_c/f_m)\eta\lambda(1 + 1/k_E)} \qquad \text{if } k_E \geq 1 \text{ and:}$$

$$k_{col} = \frac{1}{1/k_E + (f_c/f_m)\eta\lambda(1 + k_E)} \qquad \text{if } k_E \leq 1$$

making a simple design possible. The equations are slightly unsafe in the neighbourhood of $K_E = 1$ and can be corrected in the same way as done in 3.2 for lateral buckling.

Elimination of: v_0 from eq.(16) and eq.(17) gives the interaction equation:

$$\frac{M_x}{M_{ux}} = \left(1 - \frac{F}{F_c}\right)\cdot\left(1 - \frac{F\cdot F_c}{F_{ex}\cdot F_u}\right) \tag{19}$$

what is identicaly to eq.(5.1.10 d) of the Eurocode.
With the unsafe neglection of: $F\cdot F_c/(F_{ex}\cdot F_u)$ with respect to 1 in eq.(19), this equation is:

$$\frac{M_x}{M_{ux}} + \frac{F}{F_c} = 1 \tag{20}$$

being art. 4.5.4 of the Dutch timber code T.G.B. 1972.
Although the neglection is unsafe, the failure condition is in the same way too safe, and eq.(20) will give a good approximation (especially for dry, low grade, large sized timber).

3.2 Flexural-torsional buckling

For bending in the main direction without compression ($M_y = 0$; $F = 0$), eq.(15') gives the expression for lateral buckling, M_{lat}:

$$\frac{M_{lat}}{f_b\cdot W_x} + \frac{u_0\cdot\left(F_{ey}\cdot\frac{M_{lat}^2}{M_{c0}^2}\right)}{f_b\cdot W_y\cdot\left(1 - \frac{M_{lat}^2}{M_{c0}^2}\right)} = 1 \tag{21}$$

where the index 0 in M_{c0} means that $F = 0$ in the expression for M_c.
For short beams, $M_{lat} \rightarrow M_{ux} = f_b\cdot W_x$ and $M_{lat} \ll M_{c0}$. So eq.(21) is approximately:

$$\frac{M_{lat}}{M_{ux}} = 1 - \frac{u_0 \cdot F_{ey}}{M_{uy}} \cdot \frac{M_{lat}^2/M_{c0}^2}{1 - M_{lat}^2/M_{c0}^2} \approx 1 - \frac{u_0 \cdot F_{ey}}{M_{uy}} \cdot \frac{M_{lat}^2}{M_{c0}^2} \cdot (1 + M_{lat}^2/M_{c0}^2)$$

Because the second term of the last expression is small and $M_{lat} \rightarrow M_{ux}$, this can safely be approximated to:

$$M_{lat} \approx \frac{M_{ux}}{1 + \frac{u_0 \cdot F_{ey}}{M_{uy}} \cdot \frac{M_{ux}^2}{M_{c0}^2} \cdot \left(1 + \frac{M_{ux}^2}{M_{c0}^2}\right)} \qquad (22)$$

or neglecting the smallest term and using a correction factor β, eq.(22) becomes:

$$M_{lat} \approx \frac{M_{ux}}{1 + \frac{u_0 \cdot F_{ey}}{M_{uy}} \cdot \frac{M_{ux}^2}{M_{c0}^2} \cdot \beta} \qquad (22')$$

For slender beams, $M_{lat} \rightarrow M_{c0} \ll M_{ux}$. So eq.(21) is approximately:

$$\left(\frac{M_{lat}}{M_{c0}}\right)^2 = \frac{1}{1 + \frac{u_0 \cdot F_{ey}}{M_{uy}} \cdot \frac{1}{1 - M_{lat}/M_{ux}}} \approx \frac{1}{1 + \frac{u_0 \cdot F_{ey}}{M_{uy}} \cdot \left(1 + \frac{M_{c0}}{M_{ux}}\right)} \qquad (23)$$

or:

$$\frac{M_{lat}}{M_{c0}} \approx \frac{1}{1 + \frac{u_0 \cdot F_{ey}}{2 \cdot M_{uy}} \cdot \left(1 + \frac{M_{c0}}{M_{ux}}\right)} \qquad (24)$$

Eq.(24) is safe for slender beams but needs a correction factor β when applied outside the slender region and can be given like:

$$\frac{M_{lat}}{M_{c0}} \approx \frac{1}{1 + \frac{u_0 \cdot F_{ey}}{2 \cdot M_{uy}} \cdot \left(1 + \frac{M_{c0}}{M_{ux}}\right)\beta} \qquad (24')$$

Eq.(22) and eq.(24) are close together in the neighbourhood where $M_{c0} = M_{ux}$ and there eq.(22') is equal to eq.(24'). The factor β can be determerd from eq.(21) for that case. Calling: $M_{lat}/M_{c0} = M_{lat}/M_{ux} = X$ and $u_0 F'_{ey}/M_{uy} = c$, then eq.(21) is, using eq.(24'):

$$\frac{1}{1 + \beta c} + \frac{c}{(1 + \beta c)^2 - 1} \approx 1 \quad \text{or: } 2\beta^2 c = \frac{1 + \beta c}{1 + \beta c/2} \approx 1 + \beta c/2 \quad \text{giving:}$$

$$\beta \approx \frac{1}{\sqrt{2c}} + \frac{1}{8} \approx \frac{1}{\sqrt{2c}} \quad \text{and in the whole range is:}$$

$$\beta c = \sqrt{\frac{u_0 F_{ey}}{2 M_{uy}}} \cdot \sqrt{\frac{F_{ey}}{F'_{ey}}} = \sqrt{\frac{u_0 F_{ey}}{2 M_{uy}}} \cdot \frac{M_{c0}}{M_{ux}} \qquad (25)$$

The equations above may also be related to the bending strength of the standard specimen. According to the Eurocode is then: $M_{u.lat}/M_{c0} = (0{,}75)^2 = 0{,}5625$ and eq.(22') becomes:

$$M_{u,lat} \approx \frac{M_{ux}}{1+\sqrt{\frac{u_0 \cdot F_{ey}}{2\,M_{uy}} \cdot \frac{M_{u,lat}}{M_{c0}}}} = \frac{M_{ux}}{1+\sqrt{\frac{u_0 \cdot F'_{ey}}{2\,M_{uy}} \cdot (0{,}75)^2}} \qquad (22'')$$

with the specific values of F_{ey} and M_{c0} (F'_{ey} and M'_{c0}) for the specimen.

$u_0F'_{ey}/2M_{uy} = 0.5 \cdot k'_{ey}\eta\lambda_y f_c/f_m = \eta E/(8f_m)$ (being for instance 30η) for the test-specimen with a distance of the bracing of $L = 40 \cdot i_y$ ($\lambda_y = 40$).

With the notations:

$k_m = M_{c0}/\left(M_{u,lat}\left(1 + 0.56 \cdot \sqrt{\eta E/(8f_m)}\right)\right)$

$k_{ins} = M_{lat}/M_{u,lat}$,

eq.(22') and (24') are corrected to the real bending strength $M_{u,lat}$:

$$k_{ins} = \frac{1 + 0.56\sqrt{\eta E/(8f_m)}}{1 + \frac{1}{k_m} \cdot \sqrt{0.5 \cdot k_{ey}\eta\lambda_y f_c/f_m}} \leq 1 \qquad \text{for } k_m \geq 1 \qquad (26)$$

$$k_{ins} = \frac{1 + 0.56\sqrt{\eta E/(8f_m)}}{\frac{1}{k_m} + \frac{1}{2}(1 + k_m)\sqrt{0.5 \cdot k_{ey}\eta\lambda_y f_c/f_m}} \qquad \text{for } k_m \leq 1 \qquad (27)$$

u_0 or η in this last equation is unsafely taken to be zero for the Eurocode eq.(5.1.6 e). Thus for: $M_{lat} < M_{u,lat}/(1.4)^2$ is stated: $k_{inst} = M_{lat}/M_{u,lat} = M_{c0}/M_{u,lat} = k'_m$.
$M_{c0} = M_{k0} \cdot \alpha_0$, according to eq.(13) and: $M_{k0} = \sqrt{F_{ey} \cdot GI_{m0}}$. For high beams is: $EI_y \ll EI_x$, and neglecting the warping rigidity, as is possible for long beams with a rectangular cross section, $GI_{m0} = GI_t$. So:

$$M_{c0} = \sqrt{\frac{\pi^2}{L^2} \cdot \frac{h \cdot b^3}{12} \cdot E \cdot G \cdot \frac{h \cdot b^3}{3}} \cdot \alpha_0 = \frac{\pi \cdot b^3 \cdot h}{6 \cdot L} \cdot E \cdot \sqrt{G_{mean}/E_{mean}} \cdot \alpha_0$$

Because G is related to E, the mean values of the division can be taken. From this the expression of the Eurocode follows:

$$\sigma_{c0} = M_{c0}/W_x = \frac{\pi \cdot b^2}{L \cdot h} \cdot E \cdot \alpha_0 \cdot \sqrt{G_{mean}/E_{mean}} \qquad (28)$$

If the real first order bending-stress is compared with σ_{c0}, then σ_{c0} has to be replaced by $\rho \cdot \sigma_{c0}$, according to table 1. Then with $l_{ef} = L/(\rho \cdot \alpha_0)$, eq.(28) is identicaly to eq. (5.1.6 e) of the Eurocode. The factor: α_0 gives the influence of the eccentricities of the lateral loading (see eq.(13)) and ρ, the influence of the moment distribution. It can be seen from eq.(13) that the values of l_{ef} of the code are for slender beams and are not on the safe side for short beams. A more simple approach is to regard the mean moment over the middle half of the beam as mentioned before and to use directly the expressions for the eccentricity. It can be concluded that the Eurocode description of k_m and k_{ins} are not general enough and a better description is necessary.

3.3 Interaction equation for flexural-torsional buckling with compression.

If the rigidities EI_x and EI_y are mutually comparable, or when a beam is only loaded in the weak direction, the stability calculation of in plane buckling (without lateral buckling) is sufficient. For the general case of lateral buckling eq.(15') applies.
For only compression ($F = F_c$; $M_x = M_y = 0$), is eq.(15'):

$$\frac{F_c}{F_u} + \frac{F_c \cdot v_0}{M_{ux} \cdot \left(1 - \frac{F_c}{F_{ex}}\right)} + \frac{F_c \cdot u_0}{M_{uy} \cdot \left(1 - \frac{F_c}{F_{ey}}\right)} = 1 \tag{29}$$

For only bending ($F = 0$; $M_x = M_{lat}$; $M_y = 0$), is eq.(15') equal to eq.(21).
Elimination of u_0 and v_0 from: eq.(15'), eq.(21) and eq.(29) gives, with safe neglection of:

$\left(\frac{F \cdot F_c}{F_{ex} \cdot F_u} - \frac{F^2}{F_{ex} \cdot F_u}\right)$; $\left(\frac{M_x^2 \cdot M_{lat}}{M_c^2 \cdot M_{ux}} - \frac{M_x^3}{M_c^2 \cdot M_{ux}}\right)$ and $\left(\frac{F \cdot M_x}{F_{ey} \cdot M_{ux}} - \frac{F \cdot M_x^2}{F_c \cdot M_c^2}\right)$, the equation:

$$-\frac{F \cdot M_x}{F_{ey} \cdot M_{ux}} + \frac{F}{F_c} + \frac{M_x}{M_{ux}} + \left(\frac{M_x^2}{M_{lat}^2} - \frac{M_x^2}{M_{lat} \cdot M_{ux}}\right) \cdot \frac{1}{\left(1 - \frac{F}{F_t}\right) \cdot \left(1 - \frac{F}{F_{ey}}\right)} \approx 1 \tag{30}$$

For very slender beams, is: $M \leq M_{lat} \ll M_{ux}$ and is eq.(30) approximately:

$$M_x^2 \approx M_{lat}^2 \cdot \left(1 - \frac{F}{F_c}\right) \cdot \left(1 - \frac{F}{F_t}\right) \cdot \left(1 - \frac{F}{F_{ey}}\right) \tag{31}$$

Because for such beams useally: $F \leq F_c \rightarrow F_{ey} \ll F_t$ is eq.(31):

$$\frac{M_x}{M_{lat}} \approx \sqrt{\left(1 - \frac{F}{F_c}\right) \cdot \left(1 - \frac{F}{F_{ey}}\right)} \approx 1 - \frac{F}{F_c} \tag{32}$$

For short beams is: $F_{ey} \gg F_c$ and $M_{lat} \rightarrow M_{ux}$ and in eq.(30), the dominating terms are:

$$\frac{F}{F_c} + \frac{M_x}{M_{ux}} \approx 1 \tag{33}$$

Equations (32) and (33) are aproximately linear and a liear interaction equation is better than a parabolic one as often is chosen in regulations, for instance in the form of eq.(31), without the terms with F_t and F_{ey}. Also the choise of such a parabolic equation in combination with eq.(33), as is proposed in [5], can be unsafe. Better, but conservative, is to use eq.(32), that approaches eq.(33) for short beams because $M_{lat} \rightarrow M_{ux}$.
In fig. 2 some possible interaction curves are given for rectangular beams (λ_y = 25 to 150) It can be seen that this curve can be approximated by two straight lines: $y = 1 - c \cdot x$ and: $y = (1 - x)/c$. The intersecton of the lines is in the point $\left(\frac{1}{1+c}, \frac{1}{1+c}\right)$, or the point of intersection of the line: $y = x$, with the interacton curve, eq.(30). This point of intersection is dependent on the parameters: $K_m = M_{lat}/M_{ux}$ and: $K_c = F_c/F_{ey} = k_{col}/k_{eu}$ and if the expression for intersection is approached by the first terms of a row expansion in these pa-

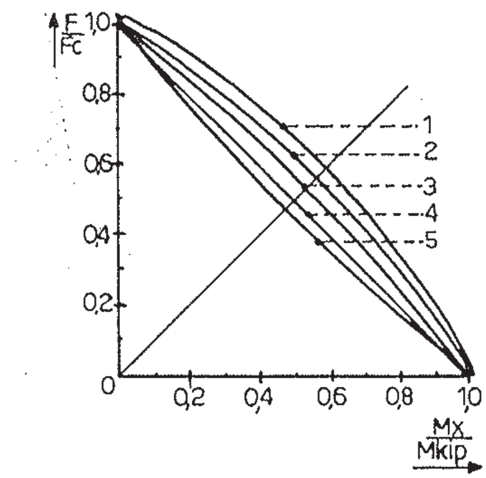


Fig. 2. Interaction curves for beams with rectangular cross section.

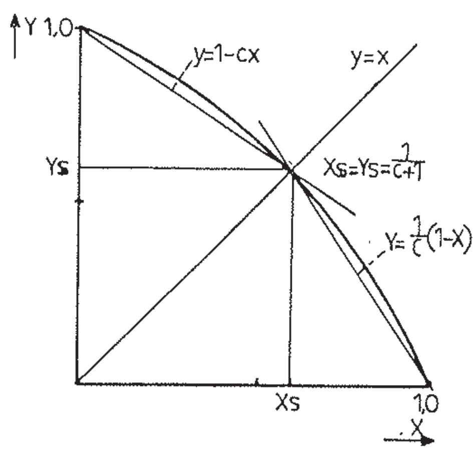


fig. 3. Approximation of the interaction curve by 2 straight lines.

rameters, then:

$$x_s = y_s = \frac{1}{1+c} = \frac{1.25 - 0.25 \cdot K_m + 2 \cdot K_m \cdot K_c}{2 + K_c} \quad \text{or: } c = \frac{2 + K_c}{1.25 - 0.25 \cdot K_m + 2 \cdot K_m \cdot K_c} - 1$$

and the interaction equations are for this value of c:

$$\frac{F}{F_c} + c \cdot \frac{M_x}{M_{lat}} \leq 1 \qquad \text{if: } \frac{M_x}{M_{lat}} \leq \frac{F}{F_c} \tag{34}$$

$$c \cdot \frac{F}{F_c} + \frac{M_x}{M_{lat}} \leq 1 \qquad \text{if: } \frac{M_x}{M_{lat}} \geq \frac{F}{F_c} \tag{35}$$

The value of F_t in eq.(30) has not much influence, so the lower bound is taken for torsionally weak beams wherefor $F_t \rightarrow F_{ey}$. Eq.(34) is, unsafe, in the code T.G.B. 1972, with: $K_{lat}/c = 2$. So: $F/F_c + M_x/2 \cdot M_{ux} \leq 1$ (art. 4.5.4).
With: c = 1, and: $M_{lat} \approx F_c \cdot z/2$, where z is the lever arm of the moment ($M_x = z \cdot N_x$), is:
$\frac{F}{2} + \frac{N_x \cdot z}{M_{lat}} \cdot \frac{F_c}{2} = \frac{F_c}{2} = N_x + \frac{F}{2}$, giving the calculation based on the buckling of the compressed flange. This condition is fullfilled for torsional weak I- beams in pure bending. For rectangular beams this calculation is safe. However for short torsional weak I-beams and trusses lateral loaded on the compressed flange, this method is unsafe.

4 Loading of the stability bracing

4.1 General equations

The loading of the lateral support by the bracing can be dtermined by eq.(2") and (3"), where the p changes sign and is equal to the loading of the bracing. At the place of the supporting bracing at height s, the deformation of the beam must be equal to the deformation of the bracing u_h. So:

$$u - u_0 + s \cdot (\varphi - \varphi_0) = u_h \tag{36}$$

Also the total load of the m supported beams: $m \cdot p$, has to be equal to the loading of the bracing. So:

$$u'''' - u_0'''' + s \cdot (\varphi'''' - \varphi_0'''') = \frac{m \cdot p + W}{EI_h} = \frac{p_h}{EI_h} \tag{37}$$

where W is the loading by the wind and EI_h is the stiffness of the bracing.
Equations (2") and (3") become (with $s_q = 0$):

$$EI_y(u'''' - u_0'''') + (M \cdot \varphi)'' + Fu'' + p = 0 \tag{38}$$

$$GI_v(\varphi'' - \varphi_0'') - M_x \cdot u'' + q \cdot e \cdot \varphi - p \cdot s = 0 \tag{39}$$

From eq.(37), eq.(38 and (39), the unknown loading p can be eliminated and two equations in u and φ remain. From the solution of these equations, p can be calculated. The solution of these two equations is analogous as for double bending as given in the apendix (see [6]). The lateral stiffness of the m supported beams is of lower order than the stiffness of the breacing, or: $m \cdot EI_y \ll EI_h$ and if, as before, the influence of φ_0 is neglected, then the loading of the bracing is:

$$p_v = \frac{\frac{m\cdot\pi^2\cdot u_0}{L^2\cdot s^2}\cdot\left[(GI_v - e\cdot M)\cdot\frac{F}{F_{eh}} + \frac{M^2}{F_{eh}}\right] + W\cdot\left(\frac{M\cdot(2\cdot s - e) + GI_v - s^2\cdot F}{s^2\cdot F_{eh}}\right)}{\left(1 - \frac{F}{F_{eh}}\right)\cdot\left(1 + \frac{GI_v - e\cdot M}{F_{eh}\cdot s^2}\right) - \left(1 - \frac{M}{F_{eh}\cdot s}\right)^2} \tag{40}$$

with: $F_{eh} = \frac{\pi^2\cdot EI_h}{m\cdot L^2}$.

4.2 Bracing and loading on the upper boundary of the lateral supported beams

For this case is: $F = 0$ and $e = s = h/2$ and eq.(40) becomes:

$$p_v = \frac{\pi^2\cdot u_0\cdot m\cdot M^2/L^2 + W\cdot(M\cdot s + GI_{v0})}{GI_{v0} + M\cdot s - M^2/F_{eh}} \tag{41}$$

For the loading of the bracing is:

$$p_v = \frac{\pi^4}{L^4}\cdot EI_h\cdot u_h \tag{42}$$

and elimination of EI_h from eq.(41) and eq.(42) gives:

$$p_v = \frac{\pi^2\cdot M^2\cdot(u_h + u_0)\cdot m}{L^2\cdot(GI_{v0} + M\cdot s)} + W \tag{43}$$

If the influence of φ_0 was not neglected eq.(43) would have been:

$$p_v = \frac{\pi^2\cdot M^2\cdot(u_h + u_0)\cdot m}{L^2\cdot(GI_{v0} + M\cdot s)} + \frac{\pi^2}{L^2}\cdot\varphi_0\cdot m\cdot M + W \tag{44}$$

Equation (43) can be written:

$$p_v = \frac{m\cdot M}{L\cdot h}\cdot\frac{20\cdot\frac{u_0 + u_v}{L}}{1 + \frac{2\cdot GI_{v0}}{M\cdot h}} + W \tag{45}$$

It is safe to take $M = M_{ux}$ in the denominator, giving a general equation for p_v. For rectangular beams is the term:

$$\frac{2\cdot GI_v}{M_{ux}\cdot h} = \frac{\bar{E}}{16}\cdot\frac{0.9\cdot h\cdot b^3\cdot 12}{3\cdot h\cdot f_b\cdot b\cdot h^2} \approx 75\cdot\frac{b^2}{h^2}$$

For common beams is: $u_0/L < \sim 1/400$ and a bracing is useally stiffer than: $u_h/L < 1/600$, and p_v is for a beam with $h = 10\cdot b$ (without wind):

$p_v \approx \frac{m\cdot M}{21\cdot L\cdot h}$, and the total force is: $P_v = \frac{2}{\pi}\cdot p_v\cdot L = 0.03\cdot\frac{m\cdot M}{h}$

This is equivalent to the value of the T.G.B.: $0{,}03 \cdot m \cdot F_{max}$ of braced bars.
For a rigid bracing is $u_v = 0$ and is $P_v \approx 0{,}02\ m \cdot M/h$, as also is given in the T.G.B.

4.3 Bracing at the center or in the tension zone of lateral supported beams

The same approach as for unbraced beams can be followed for braced beams [8] and the loading of each beam is by:

$p = p_v/m - W/m$

leading to a simular equation as eq.(11') for the second order moment. The beam has to satisfy the failure conditions leading to a simular equation as eq.(15').
The same simplifications for braced beams can be given as is done in chap. 3 for unbraced beams (see [8]) leading to comparable equations. If the bracing is in the upper part of the compression zone of the beam ($2s - e \geq -\sqrt{GI_{v0}/F_{eh}}$) the equations of 3.1 apply and the interaction equation is the same as eq.(19).
In general the system may become unstable if the bracing is at the tension zone and the loading is on the compression side or in the centre (for instance for pure bending is $e = 0$) so when $2s - e$ is negative or: $2s - e \leq -\sqrt{GI_{v0}/F_{eh}}$ (≈ 0). This instability can only be avoided by sufficient torsional stiffness of the beams thus:
Stability is provided if: $s \geq e/2$ (bracing at the compression side of the beam) or when:
$GI_{v0} \geq (1/0.75^2) \cdot M_{u,lat} \cdot (e - 2 \cdot s) = 1.75 \cdot M_{u,lat} \cdot (e - 2 \cdot s)$ where $s < e/2$, (bracing at the tension side, loading at the compression side) and M_{ux} is positive and s and e are positive when pointing from the centre to the direction of the compression side.
Because of this last requirement $h/b < \sim 4$ is necessary to have no reduced bending strength for rectangular beams with bracing at the lower tension edge of the beam.
If this requirement is not fullfilled the following design rules apply:

$$M_{lat} = \frac{M_{ux}}{1 + \sqrt{\frac{u_0 M_{ux}^2}{M_{uy} GI_{v0}} \cdot \left(1 + \frac{M_{ux}}{M_{c0}}\right)}} \qquad \text{if } M_{ux} \leq M_{c0} \tag{46}$$

$$M_{lat} = \frac{M_{c0}}{1 + \sqrt{\frac{u_0 M_{ux}^2}{M_{uy} GI_{v0}} \cdot \left(1 + \frac{M_{c0}}{M_{ux}}\right) \cdot \frac{M_{c0}^2}{M_{ux}^2}}} \qquad \text{if } M_{ux} \geq M_{c0} \tag{47}$$

with: $M_{c0} = \frac{GI_{v0}}{e - 2z}$; $e - 2z \geq \sqrt{GI_{v0}/F_{eh}}$

With the notations given before:

$k_m = M_{c0}/\left(M_{u,lat}\left(1 + 0.56 \cdot \sqrt{\eta E/(8f_m)}\right)\right)$

$k_{ins} = M_{lat}/M_{u,lat}$'

eq.(46) and (47) become:

$$k_{ins} = \frac{1 + 0.56\sqrt{\eta E/(8f_m)}}{1 + \frac{1}{\sqrt{k_m}}(1 + \frac{1}{k_m})\sqrt{\eta\lambda_y r_x/(e - 2z)}} \quad \text{for } k_m \geq 1 \qquad (46')$$

$$k_{ins} = \frac{1 + 0.56\sqrt{\eta E/(8f_m)}}{\frac{1}{k_m} + \sqrt{k_m}(1 + k_m)\sqrt{\eta\lambda_y r_x/(e - 2z)}} \quad \text{for } k_m \leq 1 \qquad (47')$$

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Appendix 1

Galerkin resolution of the differential equations.

The differential equatons (2''') and (3'''): $\bar{L}_2(u,\varphi) = 0$ and $\bar{L}_3(u,\varphi) = 0$ are solved for the first expanded of the Fourier sinus series:

$$u = \bar{u}\cdot\sin(\pi\cdot z/L);\; u_0 = \bar{u}_0\cdot\sin(\pi\cdot z/L);\; \varphi = \bar{\varphi}\cdot\sin(\pi\cdot z/L);\; \varphi_0 = \bar{\varphi}_0\cdot\sin(\pi\cdot z/L);$$
$$p = \bar{p}\cdot\sin(\pi\cdot z/L);\; M_y = \bar{M}_y\cdot\sin(\pi\cdot z/L), \text{ where } p = d^2(M_y)/dy^2.$$

For the loading in the main direction M_x, q, two known ezpanded terms are regarded in order to see the accuracy of the description by the first expanded term for special loadings. The second expanded term with $\sin(2\cdot\pi\cdot z/L)$, has no influence in this case.

$$M_x = M_1\cdot\sin(\pi\cdot z/L) + M_3\cdot\sin(3\cdot\pi\cdot z/L);\quad q = q_1\cdot\sin(\pi\cdot z/L) + q_3\cdot\sin(3\cdot\pi\cdot z/L).$$

In the Galerkin equations: $\int_0^L \bar{L}_2(u,\varphi)\cdot f_i(z)\cdot dz = 0$ and: $\int_0^L \bar{L}_3(u,\varphi)\cdot f_i(z)\cdot dz = 0$, is:

$$f_i(z) = f_1(z) = \sin(\pi\cdot z/L)$$

and these equations become:

$$\int_0^L\Big[EI_y\cdot\frac{\pi^4}{L^4}\cdot(\bar{u}-\bar{u}_0)\cdot\sin^2(\pi\cdot z/L) - F\cdot\bar{u}\cdot\frac{\pi^2}{L^2}\cdot\sin^2(\pi\cdot z/L) - \bar{\varphi}\cdot\frac{\pi^2}{L^2}\cdot\sin^2(\pi\cdot z/L)\cdot\Big\{M_1\cdot\sin(\tfrac{\pi\cdot z}{L})$$
$$+ M_3\cdot\sin(3\cdot\tfrac{\pi\cdot z}{L})\Big\} + 2\cdot\frac{\pi^2}{L^2}\cdot\bar{\varphi}\cdot\cos(\pi\cdot z/L)\cdot\sin(\pi\cdot z/L)\cdot\Big\{M_1\cdot\cos(\tfrac{\pi\cdot z}{L}) + 3\cdot M_3\cdot\cos(\tfrac{3\cdot\pi\cdot z}{L})\Big\} +$$
$$+ \bar{\varphi}\cdot\frac{\pi^2}{L^2}\cdot\sin^2(\pi\cdot z/L)\cdot\Big\{M_1\cdot\sin(\tfrac{\pi\cdot z}{L}) + 9\cdot M_3\cdot\sin(3\cdot\tfrac{\pi\cdot z}{L})\Big\} - M_y\cdot\frac{\pi^2}{L^2}\cdot\sin^2(\pi\cdot z/L)\Big]\cdot dz = 0$$

$$\int_0^L\Big[-GI_m\cdot(\bar{\varphi}-\bar{\varphi}_0)\cdot\frac{\pi^2}{L^2}\cdot\sin^2(\pi\cdot z/L) + \bar{u}\cdot\frac{\pi^2}{L^2}\cdot\sin^2(\pi\cdot z/L)\cdot\Big\{M_1\cdot\sin(\tfrac{\pi\cdot z}{L}) + M_3\cdot\sin(3\cdot\tfrac{\pi\cdot z}{L})\Big\}+$$
$$+ e_m\cdot\bar{\varphi}\cdot\sin^2(\pi\cdot z/L)\cdot\Big\{q_1\cdot\sin(\tfrac{\pi\cdot z}{L}) + q_3\cdot\sin(3\cdot\tfrac{\pi\cdot z}{L})\Big\} + \bar{p}\cdot s_m\cdot\sin^2(\pi\cdot z/L)\Big]\cdot dz = 0$$

In these equations are:

$$\frac{\pi}{L}\cdot\int_0^L \sin^2(\pi\cdot z/L)\cdot dz = \int_0^L \sin^2(\pi\cdot z/L)\cdot d(\pi\cdot z/L) = \int_0^\pi \sin^2(\alpha)\cdot d\alpha = \frac{\pi}{2}$$

$$\int_0^\pi \sin^3(\alpha)\cdot d\alpha = \frac{4}{3};\quad \int_0^\pi \sin^2(\alpha)\cdot\sin(3\alpha)\cdot d\alpha = -\frac{4}{15};\quad \int_0^\pi \sin(\alpha)\cdot\cos^2(\alpha)\cdot d\alpha = \frac{2}{3}$$

$\int_0^\pi \sin(\alpha)\cdot\cos(\alpha)\cdot\cos(3\cdot\alpha)\cdot d\alpha = -\frac{2}{5}$. So the equations become:

$$EI_y\cdot\frac{\pi^4}{L^4}\cdot(\bar{u}-\bar{u}_0)\cdot\frac{\pi}{2} - F\cdot\bar{u}\cdot\frac{\pi^2}{L^2}\cdot\frac{\pi}{2} - \bar{\varphi}\cdot\frac{\pi^2}{L^2}\cdot M_1\cdot\frac{4}{3} + \bar{\varphi}\cdot\frac{\pi^2}{L^2}\cdot M_3\cdot\frac{4}{15} + 2\cdot\frac{\pi^2}{L^2}\cdot\bar{\varphi}\cdot M_1\cdot\frac{2}{3} +$$

$$- 2\cdot\frac{\pi^2}{L^2}\cdot\bar{\varphi}\cdot 3\cdot M_3\cdot\frac{2}{5} - \bar{\varphi}\cdot\frac{\pi^2}{L^2}\cdot M_1\cdot\frac{4}{3} + 9\cdot M_3\cdot\bar{\varphi}\cdot\frac{\pi^2}{L^2}\cdot\frac{4}{15} - \bar{M}_y\cdot\frac{\pi^2}{L^2}\cdot\frac{\pi}{2} = 0$$

and:

$$- GI_m\cdot(\bar{\varphi} - \bar{\varphi}_0)\cdot\frac{\pi^2}{L^2}\cdot\frac{\pi}{2} + \bar{u}\cdot\frac{\pi^2}{L^2}\cdot M_1\cdot\frac{4}{3} - \bar{u}\cdot M_3\cdot\frac{\pi^2}{L^2}\cdot\frac{4}{15} + e_m\cdot\bar{\varphi}\cdot q_1\cdot\frac{4}{3} - e_m\cdot\bar{\varphi}\cdot q_3\cdot\frac{4}{15} +$$

$$+ \bar{p}\cdot s_m\cdot\frac{\pi}{2} = 0$$

or:

$$EI_y\cdot\frac{\pi^2}{L^2}\cdot(\bar{u} - \bar{u}_0) - F\cdot\bar{u} - \bar{\varphi}\cdot\frac{8}{3\cdot\pi}\cdot M_1 + \bar{\varphi}\cdot\frac{8}{15\cdot\pi}\cdot M_3 - \bar{M}_y = 0$$

and:

$$- GI_m\cdot(\bar{\varphi} - \bar{\varphi}_0) + \frac{8}{3\cdot\pi}\cdot M_1\cdot\bar{u} - \frac{8}{15\cdot\pi}\cdot M_3\cdot\bar{u} + e_m\cdot\bar{\varphi}\cdot q_1\cdot\frac{L^2}{\pi^2}\cdot\frac{8}{3\cdot\pi} - e_m\cdot\bar{\varphi}\cdot q_3\cdot\frac{L^2}{\pi^2}\cdot\frac{8}{15\cdot\pi} +$$

$$+ \bar{p}\cdot\frac{L^2}{\pi^2}\cdot s_m = 0$$

With: $q = \frac{8}{3\pi}\cdot\bar{q}_1\cdot\left(1 - \frac{\bar{q}_3}{5\cdot\bar{q}_1}\right)$; $M_x = \frac{8}{3\pi}\cdot M_1\cdot\left(1 - \frac{M_3}{5\cdot M_1}\right)$; $F_{ey} = \frac{\pi^2}{L^2}\cdot EI_y$; $e'_m = e_m\cdot\frac{q\cdot L^2}{\pi^2\cdot M_x}$ and:

$s'_m = s_m\cdot\frac{\bar{p}\cdot L^2/\pi^2}{\bar{M}_y}$ are these equations:

$$F_{ey}\cdot(\bar{u} - \bar{u}_0) - F\cdot\bar{u} - \bar{\varphi}\cdot M_x - \bar{M}_y = 0$$

$$GI_m\cdot(\bar{\varphi} - \bar{\varphi}_0) - M_x\cdot\bar{u} - e'_m\cdot\bar{\varphi}\cdot M_x - s'_m\cdot\bar{M}_y = 0$$

From these equations are $\bar{u}$ and $\bar{\varphi}$ solvable. So is:

$$u = \frac{\bar{M}_y\cdot(GI_m - e'_m\cdot M_x + s'_m\cdot M_x) + M_x\cdot GI_m\cdot\bar{\varphi}_0 + F_{ey}\cdot\bar{u}_0\cdot(GI_m - e'_m\cdot M_x)}{(F_{ey} - F)\cdot(GI_m - e'_m\cdot M_x) - M_x^2}$$

The total moment $M_{y,F}$ (with M_y and M_x as first order part) follows from:

$$M_{y,F} = - EI_y\cdot(u - u_0)'' \to \bar{M}_{y,F} = \frac{\pi^2}{L^2}\cdot EI_y\cdot(\bar{u} - \bar{u}_0) = F_{ey}\cdot(\bar{u} - \bar{u}_0) \to$$

$$M_{y,F} = \frac{M_y\cdot(GI_m + (s'_m - e'_m)\cdot M_x) + GI_m\cdot M_x\cdot\varphi_0 + M_x^2\cdot u_0 + F\cdot u_0\cdot(GI_m - e'_m\cdot M_x}{(F_{ey} - F)\cdot(GI_m - e'_m\cdot M_x) - M_x^2}\cdot F_{ey}$$

Appendix 2: Approximations of M_c

In general the Euler moment of lateral buckling is:

$$M_{co} = \sqrt{F_{ey}\cdot GI_{mo}}\cdot\left(\sqrt{(e_{mo}/2)^2\cdot(F_{ey}/GI_{mo})+1} - \sqrt{(e_{mo}/2)^2\cdot(F_{ey}/GI_{mo})}\right) =$$

$$= \frac{F_{ey}\cdot h}{2\left(1-\frac{EI_y}{EI_x}\right)}\cdot\left(\sqrt{\frac{e^2}{h^2}+\frac{4GI_v\left(1-EI_y/(EI_x)\right)}{h^2F_{ey}}} - \frac{e}{h}\right)$$

or:

$$\sigma_{co} = \frac{\sigma_{Eu}\cdot h/(2r_x)}{1-I_y/I_x}\cdot\left(\sqrt{\frac{e^2}{h^2}+\left(\frac{4GI_t}{h^2F_{ey}}+\frac{4C_w}{h^2I_y}\right)\cdot\left(1-\frac{I_y}{I_x}\right)} - \frac{e}{h}\right) \tag{1}$$

Only high beams need to be controlled for lateral buckling.
For high beams (e.g. $I_y < I_x/4$) eq.(1) can be simplified to:

$$\sigma_{co} = \sigma_{Eu}\cdot\frac{h}{2r_x}\cdot\left(\sqrt{\left(\frac{e}{h}\right)^2+\left(\frac{4GI_t}{h^2F_{ey}}+\frac{4C_w}{h^2I_y}\right)} - \frac{e}{h}\right) \tag{2}$$

For I-profiles and for trusses I_t has to be neglected and also for short beams the warping rigidity dominates and eq.(2) becomes:

$$\sigma_{co} = \sigma_{Eu}\cdot\frac{h}{2r_x}\cdot\left(\sqrt{\left(\frac{e}{h}\right)^2+\left(\frac{4C_w}{h^2I_y}\right)} - \frac{e}{h}\right) \tag{3}$$

or for I-beams and trusses:

$$\sigma_{co} = \sigma_{Eu}\cdot\left(\sqrt{\left(\frac{e}{h}\right)^2+1} - \frac{e}{h}\right) \tag{4}$$

For pure bending (e = 0) this becomes:

$$\sigma_{co} = \sigma_{Eu} \tag{5}$$

and for a loading at the upper edge (e = h/2) is:

$$\sigma_{co} = 0{,}62\cdot\sigma_{Eu}. \tag{6}$$

This predicted low value of lateral buckling is verified by a computer calculation of a short truss with a small lateral loading to simulate an initial displacement.

For long beams the torsional rigidity may dominate and eq.(2) becomes:

$$\sigma_{co} = \sigma_{Eu}\cdot\frac{h}{2r_x}\cdot\left(\sqrt{\left(\frac{e}{h}\right)^2+\left(\frac{4GI_t}{h^2F_{ey}}\right)} - \frac{e}{h}\right) \tag{7}$$

or for pure bending (e = 0):

$$\sigma_{co} = \sigma_{Eu} \cdot \frac{h}{2r_x} \cdot \sqrt{\frac{4GI_t}{h^2 F_{ey}}} = \frac{\pi E I_y}{L W_x} \cdot \sqrt{GI_t/(EI_y)} \tag{8}$$

or for beams with a rectangular cross section (E = 16G):

$$\sigma_{co} = \frac{\pi b^2}{Lh} \cdot \sqrt{GE} = \frac{\pi E b^2}{4Lh} \tag{9}$$

In general eq.(1) is for beams with a rectangular cross section (E/G = 16):

$$\sigma_{co} = \frac{\sigma_{Eu} \cdot 3}{1 - b^2/h^2} \cdot \left(\sqrt{\frac{e^2}{h^2} + \left(\frac{L^2}{h^2 \pi^2} + 0.3\right) \cdot \left(1 - \frac{b^2}{h^2}\right)} - \frac{e}{h}\right) \tag{10}$$

Determining is a loading at the upper edge (compression side wherefore e = h/2). For high beams (e.g. b < h/2), loaded at the upper edge (e = h/2) eq.(10) becomes:

$$\sigma_{co} = 3 \cdot \sigma_{Eu} \cdot \left(\sqrt{0{,}55 + L^2/(\pi^2 h^2)} - 0{,}5\right) \tag{11}$$

For short high beams (with dominating warping rigidity: $GI_t \ll \pi^2 EC_w/L^2$ or: L < h for beams with a rectangular cross section) in pure bending, e = 0, (e.g. the part between two lateral supports) eq.(10) becomes:

$$\sigma_{co} = 3 \cdot \sigma_{Eu} \cdot \sqrt{0.3} = 1.64 \cdot \sigma_{Eu} \tag{12}$$

For compact beams torsional instability may be determining. When $EI_y = EI_x$ lateral buckling will not occur but uncoupled buckling and/or torsional buckling is possible. In the limit case M_c becomes for $EI_y \longrightarrow EI_x$ (loading at the compression side or e > 0):

$$M_{co} = \sqrt{F_{ey} \cdot GI_{mo}} \cdot \left(\sqrt{(e_{mo}/2)^2 \cdot (F_{ey}/GI_{mo}) + 1} - \sqrt{(e_{mo}/2)^2 \cdot (F_{ey}/GI_{mo})}\right) =$$

$$= \sqrt{F_{ey} \cdot GI_{mo}} \cdot (e_{mo}/2) \cdot \sqrt{F_{ey}/GI_{mo}} \cdot \left(\sqrt{1 + 4GI_{mo}/(F_{ey} e_{mo}^2)} - 1\right) \approx$$

$$\approx \sqrt{F_{ey} \cdot GI_{mo}} \cdot (e_{mo}/2) \cdot \sqrt{F_{ey}/GI_{mo}} \cdot \left(1 + 2GI_{mo}/(F_{ey} e_{mo}^2) - 1\right) = GI_{mo}/e_{mo} =$$

$$= GI_{vo}/e = (GI_t + \pi^2 C_w/L^2)/e \tag{13}$$

When the torsional rigidity dominates is: $M_{co} \approx GI_t/e$ (torsional buckling).
When the warping rigidity dominates is: $M_{co} \approx \pi^2 C_w/(eL^2)$ (buckling of the upper flange). For I-beams and trusses loaded at the upper edge (e = h/2) this is:

$$M_{co} = \frac{\pi^2}{L^2} \cdot \frac{EI_y h^2/4}{h/2} = F_{ey} \cdot \frac{h}{2} \longrightarrow N_{fl} = \frac{M_{co}}{h} = \frac{F_{ey}}{2} = F_{ey,fl}$$

or the compresssion force in the upper flange by the Euler moment is equal to the

Euler buckling load of the upper flange. This is comparable with eq.(5) and smaller than the equivalent eq.(6).

For a beam with a square cross section the warping deformation is neglectable and also torsional buckling is not determining. The beam can only buckle in the loading direction.

For a cross beam (—|— with: $I_y = I_x$) with dominating warping rigidity and no rigid joints between the flanges is (counting only one flange):

$$\sigma_{co} = \frac{M_{co}}{bh^2/6} \approx \frac{\pi^2}{eL^2} \cdot 0{,}3 \cdot E \cdot \frac{hb^3}{12} \cdot \frac{h^2}{4} \cdot \frac{6}{bh^2} = \frac{\pi^2 Ehb^3}{L^2 12} \cdot \frac{0{,}9}{bh} = 0{,}9 \cdot \sigma_{Eu}$$

when $e = h/2$ (loading at the upper edge).

The same value is found for pure compression $\sigma_t = F_{tor}/A$:

$$\sigma_t = \frac{\pi^2}{L^2} \cdot \frac{EC_w}{I_x + I_y} \approx \frac{\pi^2}{L^2} \cdot \frac{Ehb^3/12}{bh^3/12} \cdot \frac{0.3 \cdot h^2/4}{} = \frac{\pi^2}{L^2} \cdot \frac{Ehb^3/12}{bh} \cdot 0.9 = 0.9 \cdot \sigma_{Eu}$$

Thus for flange-less profiles at the compression side (⊥, |—|, |__| profiles) instability is due to buckling of the compressed flange-less web.

Appendix 3 Proposal for design rules for the Eurocode

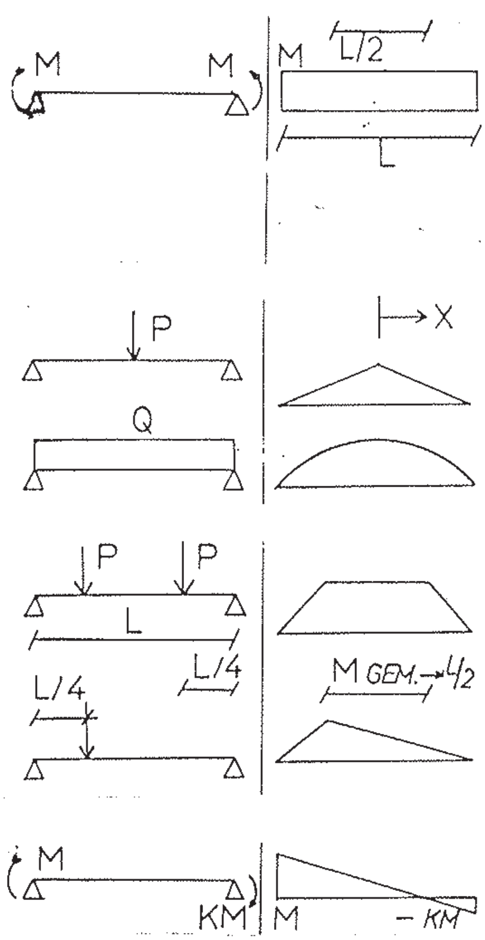
5.1.6 Bending

The following conditions shall be satisfied:

$$\sigma_{m,d} \leq k_{inst} \cdot f_{m,d} \qquad (5.1.6\ a)$$

$\sigma_{m,d}$ follows from the mean moment of the middle half of the beam with length L (see figure 5.1.6) between the supports preventing rotation and lateral displacement.

$$\sigma_{m,d} = \frac{2}{L \cdot W_y} \cdot \int_{-L/4}^{+L/4} M dx \qquad (5.1.6\ b)$$



exact value M/ρ	$M/\rho = M_{mean}$ approx.
$\frac{M}{1}$	$\frac{M}{1}$
$\frac{PL/4}{1.35}$	$\frac{PL/4}{1.33}$
$\frac{q \cdot L^2/8}{1.13}$	$\frac{qL^2/8}{1.09}$
$\frac{PL/4}{1.04}$	$\frac{PL/4}{1}$
$\frac{3PL/16}{1.44}$	$\frac{3PL/16}{1.33}$
$M \cdot (0.6 + 0.4 \cdot \kappa) \geq 0.4 \cdot M$	

Figure 5.1.6. Examples of the mean moment over the middle half of the beam.

When the beam is loaded at the supports ($x = -\frac{L}{2}$ and $x = \frac{L}{2}$) by boundary moments M and $\varkappa M$ (where $-1 \le \varkappa \le 1$) is:

$$\sigma_{m,d} = \frac{M \cdot (0,6 + 0,4 \cdot \varkappa)}{W_y} \ge \frac{0,4 \cdot M}{W_y} \qquad (5.1.6\ c)$$

The minimal value of $\sigma_{m,d}$ of $0.4 \cdot M$ applies also for combined lateral loading with boundary moments.

Cantilevers can safely be regarded as two times longer symmetrical loaded beams (having zero reactions at the ends).

k_{inst} is a factor (≤ 1) taking into account the reduced strength due to failure by lateral instability (lateral buckling and torsional buckling). k_{inst} shall be so determined that the design value of the total bending stress, taking into account the effect of initial curvature, eccentricities and the deformations developed, does not exceed $f_{m,d}$.

The strength reduction may be disregarded, i.e. $k_{inst} = 1$, if displacement and torsion are prevented at the supports and if

$$f_{m,k}/\sigma_{m,crit} \le 0.56 \qquad (5.1.6\ d)$$

where $\sigma_{m,crit}$ is the critical bending stress calculated according to the classical theory of stability.

k_{inst} may also be put equal to 1 for a beam where lateral displacement of the compression side is prevented throughout its length and where rotation is prevented at the supports.

Under the assumption of an initial lateral deviaton from straightness of less than 1/300 k_{inst} may be determined from (5.1.6 e-f).

$$k_{inst} = \frac{k_m \cdot \left(1 + 0,56 \cdot \sqrt{\eta \cdot E_{0,k}/(8 \cdot f_{m,k})}\right)}{k_m + \sqrt{0,5 \cdot k_{Ez} \cdot \eta \cdot \lambda_z \cdot f_{c,k}/f_{m,k}}} \quad \text{for } k_m \ge 1 \qquad (5.1.6\ e)$$

$$k_{inst} = \frac{k_m \cdot \left(1 + 0,56 \cdot \sqrt{\eta \cdot E_{0,k}/(8 \cdot f_{m,k})}\right)}{1 + 0,5 \cdot (k_m + k_m^2) \cdot \sqrt{0,5 \cdot k_{Ez} \cdot \eta \cdot \lambda_z \cdot f_{c,k}/f_{m,k}}} \quad \text{for } k_m < 1 \qquad (5.1.6\ f)$$

In (5.1.6 e-f) is the loading in z-direction, being also the direction of the the weak axis and is:

$$\eta = \frac{i_z}{300 \cdot r_z} \qquad (5.1.6\ g)$$

where r_z is the core radius of the compression side, giving for solid timber:

$$\eta = \frac{b}{\sqrt{12}} \cdot \frac{6}{b} \cdot \frac{1}{300} = 0.006, \qquad (5.1.6\ h)$$

$$\lambda_z = \frac{l_{ef}}{i_z} \qquad (5.1.6\ i)$$

where l_{ef} is the buckling length and $i_z = \sqrt{I_z/A}$,

$$k_{Ez} = \frac{\sigma_{Ez}}{f_{c;0;rep}} = \frac{\pi^2 \cdot E_{0,k}}{\lambda_z^2 \cdot f_{c,k}} \qquad (5.1.6\ j)$$

$$k_m = \frac{\sigma_{m,crit}}{f_{m,k} \cdot \left(1 + 0{,}56 \cdot \sqrt{\eta \cdot E_{0,k}/(8 \cdot f_{m,k})}\right)} \qquad (5.1.6\ k)$$

with

$$\sigma_{m,crit} = \frac{k_{Ez} \cdot h}{2 \cdot r_y} \cdot \frac{f_{c,k}}{1 - \frac{I_z}{I_y}} \cdot \left(\sqrt{\left(\frac{e}{h}\right)^2 + \left(1 - \frac{I_z}{I_y}\right) \cdot \left(\frac{4 \cdot l_{ef}^2 \cdot G_{mean} I_{tor}}{\pi^2 \cdot E_{0,mean} I_z \cdot h^2} + \frac{4 \cdot C_w}{h^2 \cdot I_z}\right)} - \frac{e}{h}\right) \qquad (5.1.6\ l)$$

where

C_w = a warping factor: $C_w = I_z h^2/4$ for I-profiles and $C_w = 0{,}3 . I_z h^2/4 = h^3 \cdot b^3/160$ for solid timber,

e = eccentricity of the lateral loading, being the distance of the load with respect to the neutral axis if the beam is only lateraly loaded. The sign of e is positive in the direction of the compression side or,

e = h/2 for a symmetrical beam, only lateraly loaded at the upper compression side,

e = - h/2 for only lateral loading at the tensional boundary and

e = 0 for lateral loading at the neutral axis of the beam or for loading by moments at the supports (no lateral loading).

For combined lateral loading with moments at the supports e has to be replaced by e' according to:

$$e' = \frac{e}{1 + \frac{\sigma_m}{\sigma_q}} \qquad (5.1.6\ m)$$

where σ_q is the design value of the bending stress by the mean moment of the middle half of the beam by only lateral loading (eq.(5.1.6 b) and σ_m is the design value of the bending moments at the supports (eq.(5.1.6. c). When $I_z = I_y$, $\sigma_{m,crit}$ turns to the torsional value of:

$$\sigma_{m;cr} = \left(\frac{G_{mean} \cdot I_{tor}}{E_{0,mean} \cdot e \cdot W_y} + \frac{\pi^2 \cdot C_w}{l_{ef}^2 \cdot e \cdot W_y}\right) \cdot E_{0,k} \qquad (5.1.6\ n)$$

Simplifications of $\sigma_{m,crit}$ at the safe side are possible.

For solid beams (with $G_{mean}/E_{0,mean} = 1/16$) (5.1.6 l) becomes:

$$\sigma_{m,crit} = \frac{3\cdot k_{Ez}}{1-\frac{b^2}{h^2}}\cdot f_{c,k}\cdot\left[\sqrt{\left(\frac{e}{h}\right)^2+\left(1-\frac{b^2}{h^2}\right)\cdot\left(\frac{l_{ef}^2}{\pi^2\cdot h^2}+0{,}3\right)}-\frac{e}{h}\right] \tag{5.1.6 o}$$

For high solid beams (for instance if b < h/2) loaded by only bending (e = 0) this is safely:

$$= \frac{\pi\cdot b^2}{h\cdot l_{ef}}\cdot E_{0,k}\cdot\sqrt{\frac{1}{16}+\frac{\pi^2}{l_{ef}^2}\cdot 0{,}3\cdot\frac{h^2}{16}} \tag{5.1.6 p}$$

For relatively long, high, solid beams loaded by bending alone (e = 0) this may safely be replaced by:

$$\sigma_{m,crit} = \frac{\pi\cdot b^2}{h\cdot l_{ef}}\cdot E_{0,k}\cdot\frac{1}{4} \tag{5.1.6 q}$$

For relatively short, high, solid beams loaded by bending alone (e = 0) the warping rigidity dominates and may safely applied:

$$\sigma_{m,crit} = 1{,}64\cdot k_{Ez}\cdot f_{c,k} \tag{5.1.6 r}$$

For relatively high, long, solid beams lateraly loaded at the upper compression side (e = h/2) is safely:

$$\sigma_{m,crit} = 3\cdot k_{Ez}\cdot f_{c,k}\cdot\left(\sqrt{0{,}55+l_{ef}^2\;/(\pi^2\cdot h^2)}-0{,}5\right) \tag{5.1.6 s}$$

For trusses and thin-webbed profiles with a dominating warping rigidity the torsional rigidity I_{tor} in (5.1.6 l) should be neglected and is:

$$\sigma_{m;cr} = \frac{k_{Ez}\cdot h}{2\cdot r_y}\cdot\frac{f_{c;0;rep}}{1-\frac{I_z}{I_y}}\left(\sqrt{\left(\frac{e}{h}\right)^2+\left(1-\frac{I_z}{I_y}\right)\cdot\left(\frac{4\cdot C_w}{h^2\cdot I_z}\right)}-\frac{e}{h}\right) \tag{5.1.6 t}$$

For high profiles with with flanges in the compression zone, whereby these flanges mainly determine the rigidity I_z (for instance I - and T - profiles and trusses) this is:

$$\sigma_{m,crit} = k_{Ez}\cdot f_{c,k}\cdot\left(\sqrt{\frac{e^2}{2}+1}-\frac{e}{h}\right) \tag{5.1.6 u}$$

For thin high profiles with a low warping rigidity (when I_z of the compressed zone, being the web, is much smaller dan I_z of the total beam, for instance for +, ⊢⊣ - or ⊥ - profiles, torsional instability may become determining and the calculation can be based on the torsional or warping rigidity of this web and is for the lateral loading at the compression side and dominating warping rigidity:

$$\sigma_{m,crit} = 0{,}9\cdot k'_{Ez}\cdot f_{c,k} \tag{5.1.6 v}$$

where k'_{Ez} is the value of k_{Ez} when only the compressed web of the profile is counted for the rigidity.

For dominating torsional rigidity of the web the equation for long solid beams can be applied.

With respect to torsional buckling by compression alone is also:

$$\sigma_{c;0;d} < 0{,}9 \cdot k'_{Ez} \cdot f_{c;0;d} \qquad (5.1.6\ w)$$

and k_{Ez} has to be replaced by $0{,}9 \cdot k'_{Ez}$ in the expressions for columns of 5.1.10.

5.1.10 Columns

The bending stresses due to initial curvature and induced deflection shall be taken into account, in addition to those due to any lateral load.

The theory of linear elasticity may be used to calculate the resultant bending moment.

For the initial curvature a sinusiodal form may be assumed corresponding to a maximum eccentricity of the axial force of:

$$u_0 = \eta r \lambda \qquad (5.1.10\ a)$$

where r is the core radius.

For solid timber η shall as a minimum be taken as:

$$\eta = 0.006 \qquad (5.1.10\ b)$$

(corresponding for a rectangular cross-section to an initial eccentricity of about 1/300 of the length), and for glued laminated timber:

$$\eta = 0.004 \qquad (5.1.10\ c)$$

The stresses should satisfy the following conditions:

$$\frac{\sigma_{c,0,d}}{k_c \cdot f_{c,0,d}} + \frac{\sigma_{m,d}}{k_{inst} \cdot f_{m,d}} \cdot k_{mc} \leq 1 \quad \text{for} \quad \frac{\sigma_{c,0,d}}{k_c \cdot f_{c,0,d}} \geq \frac{\sigma_{m,d}}{k_{inst} \cdot f_{m,d}} \qquad (5.1.10\ d)$$

$$\frac{\sigma_{c,0,d}}{k_c \cdot f_{c,0,d}} \cdot k_{mc} + \frac{\sigma_{m,d}}{k_{inst} \cdot f_{m,d}} \leq 1 \quad \text{for} \quad \frac{\sigma_{c,0,d}}{k_c \cdot f_{c,0,d}} \leq \frac{\sigma_{m,d}}{k_{inst} \cdot f_{m,d}} \qquad (5.1.10\ e)$$

where:

$$k_{mc} = \frac{2 + k_c/k_{Ez}}{1{,}25 - 0{,}25 \cdot k_{inst} + 2 \cdot k_{inst} \cdot k_c/k_{Ez}} - 1, \qquad (5.1.10\ f)$$

$$k_c = 0{,}5 \cdot (1 + 20\eta) \cdot \left(\xi - \sqrt{\xi^2 - \frac{4 \cdot k_{Ez}}{1 + 20 \cdot \eta}}\right) \qquad (5.1.10\ g)$$

with:

$$\xi = 1 + \left(1 + \eta \cdot \lambda_z \cdot (1 + 20 \cdot \eta) \cdot \frac{f_{c,o,k}}{f_{m,k}}\right) \cdot \frac{k_{Ez}}{1 + 20 \cdot \eta} \qquad (5.1.10\ h)$$

If $\lambda_z < \lambda_y$ the equations of 5.2.6 apply for instability in the z-direction (direction of the loading) only.

5.2.6 Bracing

Adequate bracing shall be provided to avoid lateral instability of individual members and the collapse of the whole structure due to external loading such as wind.

The stresses due to initial curvature and induced deflection shall be taken into account.

The theory of linear elasticity may be used.

The initial deviations from straightness at midspan shall as a minimum be taken as 1/450 for glued laminated beams and as 1/300 for other structures.

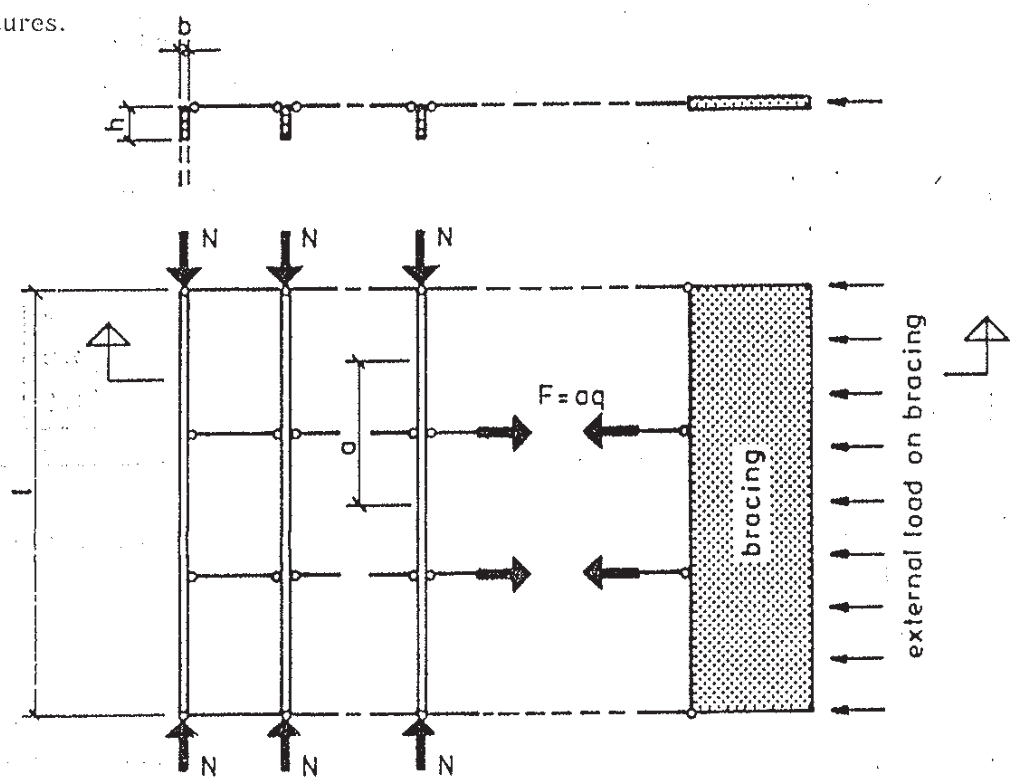


figure 5.2.6 a. Bracing system

With n equal members (e.g. beams or chords in a truss) of rectangular cross-section, the bracing should in addition to external loads (e.g. wind) be designed for a uniform load q per unit length:

$$q_{br,d} = \frac{n \cdot N_{c,d}}{l \cdot k_{br,c} \cdot k_{br,m}}$$

where

$$k_{br,m} = \frac{9 \cdot \pi}{32} \cdot (1 + 1,5\ \pi \cdot (1 - 0,63 \frac{b}{h})\cdot(\frac{b}{h})^2 \cdot \frac{E_{0,k} \cdot G_{mean}}{f_{m,k} \cdot E_{0,mean}},$$

for beams with rectangular cross sections, and:

$k_{br,m} = 1$ for compression gords of trusses.

$$k_{br,c} = \frac{l}{8 \cdot (k_n \cdot k_l \cdot u_0 + u_{br})}$$

$N_{c,d}$ is the design value or the axial force in the member.
Where the member is a beam with a rectangular cross section with a maximum moment M_d and depth h, $N_{c,d}$ should be taken as $N_{c,d} = 1.5 \cdot M_d/h$.
Where the member is a truss $N_{c,d}$ is the maximum compressive force.

$k_l = \sqrt{\frac{15}{l}} \le 1$, (where l is the span in m).

$$k_n = 0,5(1 + \frac{1}{n})$$

u_0 is the initial deviation from sthaightness at midspan

u_{br} is the deflection of the bracing caused by the sum of q and the external loads calculated with

$$E = E_{0,k} \cdot f_{m,d}/f_{m,d}$$

By the calculation of u_{br} the effect of slip in the joints should be taken into account.

Stability of the braced beams

When the bracing is connected to the members at the compression side or at the neutral axis of the lateral supported members the stresses of these members should satisfy the following condition:

$$\frac{\sigma_{c,o,d}}{k_c \cdot f_{c,o,d}} + \frac{\sigma_{m,d}}{k_{mom} \cdot f_{c,o,d}} \le 1$$

where

$$k_c = 0,5 \cdot (1 + 20\eta) \cdot \left(\xi - \sqrt{\xi^2 - \frac{4 \cdot k_{Ey}}{1 + 20 \cdot \eta}}\right) \le 1$$

$$\xi = 1 + \left(1 + \eta \cdot \lambda_y \cdot (1 + 20 \cdot \eta) \cdot \frac{f_{c,o,k}}{f_{m,k}}\right) \cdot \frac{k_{Ey}}{1 + 20 \cdot \eta}$$

$$k_{mom} = 1 - \frac{k_c}{k_{Ey}} \cdot \frac{\sigma_{c,o,d}}{f_{c,o,d}}$$

When the bracing is connected to the tension side of the lateral supported members k_{inst} is:

$$k_{inst} = \frac{1 + 0.56\sqrt{\eta E_{0,k}/(8f_{m,k})}}{1 + \frac{1}{\sqrt{k_m}}(1 + \frac{1}{k_m})\sqrt{\eta\lambda_z r_y/(e - 2z)}} \qquad \text{for } k_m \geq 1$$

$$k_{inst} = \frac{1 + 0.56\sqrt{\eta E_{0,k}/(8f_{m,k})}}{\frac{1}{k_m} + \sqrt{k_m}(1 + k_m)\sqrt{\eta\lambda_z r_y/(e - 2z)}} \qquad \text{for } k_m < 1$$

with k_m according to 5.1.6,

e the excentricity of the lateral loading according to 5.1.6,

z the height of the connection of the bracing with the same sign convention as for e.